# Capacity Recovery of Very Noisy Optical Quantum Channels

Laszlo Gyongyosi, Sandor Imre

***Abstract*****—** The number of efficient approximation algorithms for quantum informational distances is very small, because of the special properties of quantum informational generator functions and of asymmetric quantum informational distances. If we wish to analyze the properties of quantum channels using today's classical computer architectures, an extremely efficient algorithm is needed. The capacity recovery of very noisy communication channels cannot be imagined for classical systems, and this effect has no analogue in classical systems. The capacity recovery of very noisy quantum channels makes it possible to use two very noisy optical-fiber based quantum channels with a positive joint capacity at the output. Here we define a fundamentally new approach of capacity recovery of very noisy, practically completely useless optical quantum channels. We show an algorithmic solution to the capacity recovery problem, and provide an efficient algorithmic solution for finding the set of recoverable very noisy optical quantum channels. The calculations are based on the asymptotic classical capacity of the quantum channel.

***Keywords*****—** Capacity Recovery, Noisy Optical Communications, Quantum Channels, Quantum Communications.

## I. Introduction

THE capacity recovery of very noisy, practically completely useless quantum channels makes it possible to use two very noisy quantum channels with a positive joint capacity at the output. As derived in [1], two very noisy communication links can be used to transmit information through the quantum communication channel and it is possible to "activate" one channel with the other channel, so the capacity of very noisy quantum channels can be increased [1]. The process of information transmission through an optical quantum communication channel can be described in three phases. In the first phase, the sender has to encode his information, according to properties of the physical channel - this step is called *source encoding*. After the sender has encoded his information into the appropriate form, it has to put on the optical quantum channel, which transforms it according to its channel map - this second phase is called the *channel evolution*. The optical quantum channel conveys the quantum state to the receiver, however this state is still a superposed quantum state. To extract the information which is encoded in the state, the receiver has to make a measurement - this *measurement process* is the third phase of the communication over a quantum channel [2], [8], [17].

In Fig. 1, we illustrate the source coding phase. The sender encodes his information into a physical attribute of a physical particle, such as the spin of the particles. For example, in the case of an electron or a half-spin particle, this can be an axis spin. The half-spin particles could take two possible states, hence these particles are two-level quantum systems - hence qubits.

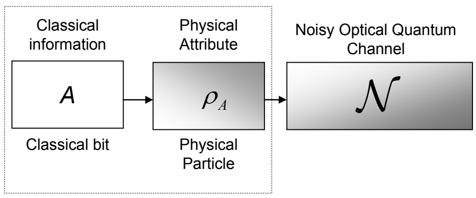


Fig. 1 The source coding phase

The channel transformation represents the noise of the quantum channel. Physically, the optical quantum channel is the medium, which moves the particle from the sender to the receiver. The noise disturbs the state of the particle, in the case of a half-spin particle, it causes spin precession. For a noisy optical quantum channel, the channel transforms the original state into a mixed state [2]. The channel evolution phase is illustrated in Fig. 2.

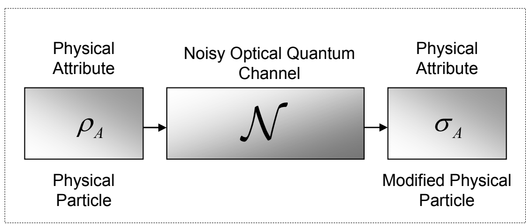


Fig. 2 The channel evolution phase

Finally, the measurement process responsible for the decoding, or the extraction of the encoded information. The previous phase determines the success probability of the recovery of the original data. If the channel is a completely noisy channel, then the receiver will get a maximally mixed quantum state - which state is geometrically positioned at the

Manuscript received January 11, 2011: Revised version received March 23, 2011.

L. Gyongyosi is with the Department of Telecommunications, Budapest University of Technology, 1521 Budapest, Hungary (e-mail: gyongyosi@hit.bme.hu).

S. Imre is with the Department of Telecommunications, Budapest University of Technology, 1521 Budapest, Hungary (e-mail: imre@hit.bme.hu).

origin of the Bloch sphere. The output of the measurement of a maximally mixed state is completely undeterministic: it tells us nothing about the original information encoded by the sender [2], [8], [38]-[42].

A general quantum channel transforms the original pure quantum state into a mixed quantum state, - but not into a maximally mixed state - which makes it possible to recover the original message with a high - or low - probability, according to the level of the noise of the quantum channel. The measurement phase is illustrated in Fig. 3.

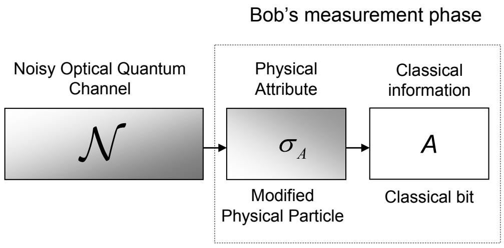


Fig. 3 The measurement process

We present an algorithmic solution to the capacity recovery problem of very noisy optical quantum channels. Currently, we have no theoretical results for describing all possible combinations of recoverable very noisy channels, hence there should be many other possible combinations. We analyze the capacity recovery of the amplitude damping channel, which is an important channel in optical physical implementations. This channel describes the effect of energy dissipation of the quantum states. In practical optical or quantum communications, where quantum states or quantum bits are used, the loss of energy from the quantum system causes amplitude damping. In many practical applications, energy dissipation is an unavoidable phenomenon, hence analysis of the amplitude damping quantum channel is a relevant issue [2], [8], [17].

### A. *Quantum Informational Distance*

In our work, we apply computational geometry in quantum space, between pure and mixed quantum states. In Fig. 4, we illustrate the logical structure of the analysis and the cooperation of classical and quantum systems. Since, currently we have no quantum computers, we would like to find recoverable noisy optical quantum channels using current classical computer architectures and the most efficient currently available algorithms. To this day, the most efficient classical algorithms for this purpose are computational geometric methods.

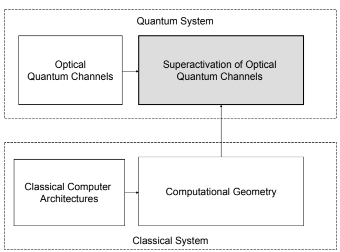


Fig. 4 The logical structure of our analysis

Unlike ordinary geometric distances, the quantum informational distance is not a metric and is not symmetric, hence this pseudo-distance features as a measure of informational distance [26], [27], [28], [29], [30], [31], [33], [34].

## II. Capacity of a Noisy Quantum Channel

The input of a quantum channel $\mathcal{N}$ can be a pure or a mixed quantum state. In the case of a *pure* quantum state, the state vector $|\psi\rangle$ of the quantum state is *completely known*. The state vector $|\psi\rangle$ is a unit vector in the state space of the quantum system, and the density matrix of this *pure* state can be expressed by the projection

$$\rho = |\psi\rangle\langle\psi| . \tag{1}$$

On the other hand, for a *mixed* quantum state the state vector is *not completely known*, and the system is one of a number of possible states. In the case of a mixed quantum system, the system is in one of the states $|\psi_i\rangle$ with a given probability $p_i$, and it forms an *ensemble* of *pure* states as $\{p_i, |\psi_i\rangle\}$.

### A. *Channel Input and Channel Output Quantum States*

The density operator of a mixed state differs from the density operator of a pure state, for a mixed state it can be expressed as $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$.

The transmission of information through the quantum channel $\mathcal{N}$ is sent through in the form of encoded quantum states. In general, the input quantum state $\rho_i$ is prepared with probability $p_i$, which describes the ensemble $\{p_i, \rho_i\}$.

The average of the *input* of the quantum states is expressed as

$$\sigma = \sum_i p_i \rho_i , \tag{2}$$

The average of the *output* of the quantum channel is denoted by

$$\mathcal{N}(\sigma) = \sum_i p_i \mathcal{N}(\rho_i) . \tag{3}$$

The classical information which can be transmitted through a noisy quantum channel $\mathcal{N}$ can be expressed by the $\chi$ Holevo quantity. The Holevo quantity describes the amount of information, which can be extracted from the output about the input state. We note, this information is also can be referred as *accessible information* in the literature. We can say, that the Holevo quantity of a quantum channel $\mathcal{N}$ expresses the joint entropy of the composite state $\rho_{in} \otimes \rho_E$.

### B. *Qubit Representations*

On the other hand, a quantum state can be described by its density matrix $\rho \in \mathbb{C}^{d\times d}$, which is a $d \times d$ matrix, where *d* is the level of the given quantum system. For an *n* qubit system, the level of the quantum system is $d = 2^n$. We use the fact that particle state distributions can be analyzed probabilistically by means of density matrices. A two-level quantum system can be defined by its density matrices in the following way:

$$\rho = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix},\ x^2+y^2+z^2 \le 1, \tag{4}$$

which also an be rewritten as

$$\rho = \begin{pmatrix} \frac{1+z}{2} & \frac{x-iy}{2} \\ \frac{x+iy}{2} & \frac{1-z}{2} \end{pmatrix},\ x^2+y^2+z^2 \le 1,\ x,y,z \in \mathbb{R}., \tag{5}$$

where $i$ denotes the complex imaginary $i^2 = -1$. The density matrix $\rho = \rho(x,y,z)$ can be identified with a *point* $(x,y,z)$ in 3-dimensional space, and a ball $\mathbf{B}$ formed by such points

$$\mathbf{B} = \left\{ (x,y,z) \middle| x^2+y^2+z^2 \le 1 \right\}, \tag{6}$$

is called a Bloch-ball. The eigenvalues $\lambda_1, \lambda_2$ of $\rho(x,y,z)$ are given by

$$\lambda_1, \lambda_2 = \left(1 \pm \sqrt{x^2+y^2+z^2}\right) \Big/ 2, \tag{7}$$

the eigenvalue decomposition $\rho$ is

$$\rho = \sum_i \lambda_i E_i, \tag{8}$$

where $E_i E_j$ is $E_i$ for $i = j$ and 0 for $i \ne j$. For a mixed state $\rho(x,y,z)$, $\log\rho$ defined by

$$\log\rho = \sum_i (\log\lambda_i) E_i. \tag{9}$$

Using the Bloch sphere representation, the quantum state $\rho$ can be given as a three-dimensional point $\rho = (x,y,z)$ in $\mathbb{R}^3$, and it can be represented in spherical coordinates

$$\rho = (r, \theta, \varphi), \tag{10}$$

where $r$ is the radius of the quantum state to the origin, $\theta$ and $\varphi$ represents the latitude and longitude rotation angles [2], [14]. Using the spherical coordinates, a three-dimensional point on the Bloch sphere can be given by:

$$\begin{aligned} x &= r\sin\theta\cos\varphi, \\ y &= r\sin\varphi, \\ z &= r\cos\theta\cos\varphi. \end{aligned} \tag{11}$$

The Bloch vectors are real 3-dimensional vectors, that have magnitude $m = 1$ for pure states, and $m < 1$ for mixed states. The Bloch vectors of the states denoted by $\mathbf{r}$, and it can be expressed as

$$\mathbf{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}. \tag{12}$$

A qubit can be described by the two-dimensional Hilbert space $\mathbb{C}^2$, and the operators acting on the quantum system is generated by the Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

For a Pauli matrix $\sigma_k$, $Tr(\sigma_k) = 0$ and $\sigma_k^2 = I$, where $k = x,y,z$. The set of states for a qubit in the computational basis $\{|0\rangle, |1\rangle\}$, is the eigenbasis of $\sigma_z$, thus $\sigma_z|0\rangle = |0\rangle$ and $\sigma_z|1\rangle = -|1\rangle$. A generic *pure* state can be given by

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{14}$$

and the *projector* of the state is $|\psi\rangle\langle\psi| = \frac{1}{2}(\mathbf{1} + \hat{n}\cdot\vec{\sigma})$, where $\hat{n}$ is the *Bloch vector*, and it can be given by $\hat{n} = \left(2\operatorname{Re}(\alpha\beta^*), 2\operatorname{Im}(\alpha\beta^*), |\alpha|^2 - |\beta|^2\right)$. For *pure* state the *norm* of Bloch vector is 1, and these vectors cover the Bloch sphere. The *pure* quantum states can be given by unit vectors in spherical coordinates

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle. \tag{15}$$

The state $|\psi\rangle$ can be given by state $|+\hat{n}\rangle$, and it is the eigenstate for the eigenvalue *+1* of $\hat{n}\cdot\vec{\sigma}$, with

$$\hat{n} = \hat{n}(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \tag{16}$$

where $\theta \in [0,\pi]$ and $\varphi \in [0,2\pi]$. A $\rho$ *mixed* state also can be expressed by a

$$|\psi\rangle\langle\psi| = \frac{1}{2}(\mathbf{1} + \hat{n}\cdot\vec{\sigma}) \tag{17}$$

projector on a *pure* quantum state.

### C. *The quantum channel model*

In the most general view of the quantum communication model, Alice's pure quantum state can be expressed by a density matrix $\rho_A$, whose rank is one, while a state with rank two is called *mixed*. According to the noise $\mathcal{N}$ of quantum channel, Alice's sent pure quantum state $\rho_A$ becomes a mixed state, thus Bob will receive a mixed state denoted by $\sigma_B$. A pure state has special meaning in quantum information theory and it is on the boundary of the convex object.

For one-qubit states, the condition for $\rho$ to be pure is simply expressed as $x^2+y^2+z^2 = 1$, and it is on the surface of the Bloch ball. The map of the quantum channel is a trace-preserving and completely positive map, and it can be given by a linear transform $\mathcal{N}$ which maps quantum states to quantum states. The classical data encoded into a quantum state represented by the density operator.

The quantum state is evolved via the quantum channel map $\mathcal{N}$, and if there is no noise on the channel, the map $\mathcal{N}$ is identical to identity transformation. The map of the channel can be modeled by a linear transform thus, if Alice sends quantum state $\rho(x,y,z)$ on the quantum channel, the channel maps it as follows:

$$\left\{ (x',y',z') \middle| \rho'(x',y',z') = \mathcal{N}(\rho(x,y,z)), (x,y,z) \right\}. \tag{18}$$

The quantum channel $\mathcal{N}$ maps the density operators from a Hilbert space to another Hilbert space. According to the noise of the quantum channel, the pure input states become mixed states, which means that the output of the quantum channel cannot be determined with absolute certainty.

The information transmission through the quantum channel $\mathcal{N}$ is defined by the $\rho_{in}$ input quantum state and the initial state of the environment, $\rho_E = |0\rangle\langle 0|$. In the initial phase, the environment is assumed to be in the pure state $|0\rangle$. The system state which consist of the input quantum state $\rho_{in}$ and the environment $\rho_E = |0\rangle\langle 0|$, is called the composite state $\rho_{in} \otimes \rho_E$.

If the quantum channel $\mathcal{N}$ is used for information transmission, then the state of the composite system changes unitarily, as follows:

$$U\left(\rho_{in} \otimes \rho_E\right)U^*, \tag{19}$$

where $U$ is a unitary transformation.

After the quantum state transmitted he quantum channel $\mathcal{N}$, the $\rho_{out}$ output state can be expressed as:

$$\rho_{out} = Tr_E\left[U\left(\rho_{in} \otimes \rho_E\right)U^*\right], \tag{20}$$

where $Tr_E$ is the partial trace operator, which traces out the environment from the joint state.

The transmission of classical information over *quantum channel* with no prior entanglement between the sender (Alice) and the recipient (Bob) is illustrated in Fig. 5. The sender's classical information denoted by $A_i$ encoded into a quantum state $|\psi_A\rangle$. The encoded quantum states are sent over the quantum channel. In the decoding phase, Bob measures state $|\psi_A\rangle$, the outcome of the measurement of his received state $|\psi_B\rangle$ is the classical information $B_i$. The classical information is illustrated by the dashed line, the solid line represents quantum information.

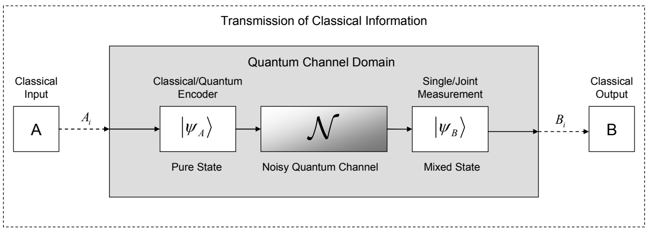


Fig. 5 Transmission of classical information through the quantum channel

In the classical communication model, the sender and receiver can be modeled by random variables

$$X = \left\{p_i = P\left(x_i\right)\right\}, i = 1, \ldots N, \tag{21}$$

and

$$Y = \left\{p_i = P\left(y_i\right)\right\}, i = 1, \ldots N. \tag{22}$$

In classical systems, the Shannon entropy of the discrete random variable $X$ is denoted by $H\left(X\right)$ and can be defined as

$$H\left(X\right) = -\sum_{i=1}^{N} p_i \log\left(p_i\right). \tag{23}$$

For conditional random variables, the probability of random variable $X$ given $Y$ is denoted by $p\left(X|Y\right)$. The noise in the channel increases the uncertainty in $X$, given Bob's output $Y$. The informational theoretic noise of the channel increases the conditional Shannon entropy $H\left(X|Y\right)$, defined as

$$H\left(X|Y\right) = \sum_{i=1}^{N_X}\sum_{j=1}^{N_Y} p\left(x_i, y_j\right)\log p\left(x_i|y_j\right), \tag{24}$$

thus the radius of the smallest enclosing quantum informational ball will decrease for fixed $H\left(X\right)$.

The general classical informational theoretic model for a noisy quantum channel is illustrated in Fig. 6. Alice's pure state is denoted by $\rho_A$, the noise is modeled by an affine map $\mathcal{N}$ and Bob's mixed input state is denoted by $\mathcal{N}\left(\rho_A\right) = \sigma_B$.

For random variables $X$ and $Y$, the mutual entropy can be expressed as

$$H\left(X,Y\right) = H\left(X\right) + H\left(Y|X\right), \tag{25}$$

where $H(X)$, $H\left(X,Y\right)$ and $H\left(Y|X\right)$ are defined by probability distributions.

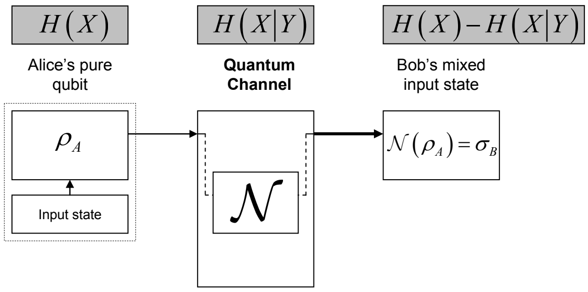


Fig. 6 The classical communication model

In the classical communication model, we seek to maximize $H\left(X\right)$ and minimize $H\left(X|Y\right)$ in order to maximize the radius of the smallest enclosing ball of Bob, since the radius can be computed as

$$r^* = \max_{\{all\ possible\ x_i\}} H\left(X\right) - H\left(X|Y\right). \tag{26}$$

Geometrically, the presence of noise on the quantum channel causes a detectable mapping to change from a noiseless one-to-one relationship to a stochastic map. In the classical model of a quantum channel, the input is in a pure state denoted by $\left\{p_i, \rho_i\right\}$ and a measurement is made at the end of the quantum channel, which extracts the classical information from the sent quantum state.

## III. Noisy Quantum Channel Coding

To describe the capacity of the quantum channel, we have to make a distinction between the various capacities of a quantum channel. The results of quantum information theory inspired from classical information theory, however not all of these classical results can be used in the quantum communication model. An other important difference between the capacities of the quantum channel, and the capacity of a classical channel, that those capacities of a quantum channel can be determined only *asymptotically*, which makes the case more hard computationally. It means, that while the capacity of a classical communication channel can be determined by the *single-use* or *one-shot* method, in the case of the quantum channel, the capacities can be determined only *asymptotically*, - or with other word, the one-shot capacity is not equal to the capacity of the quantum channel. This fundamental difference make possible to use the quantum channel for information transmission in those situations, which are completely unimaginable in the case of a classical communication channel [32].

### A. Classical Channel Coding

Before we start to discuss the method of quantum channel encoding, we introduce the subject by the results of the classical noisy channel coding. The classical communication channel $N$ encodes the information into classical information carrier, and transmits classical bits. The capacity of a classical communication channel $N$ (which channel does not use quantum states for encoding) gives an upper bound on the classical bits which can be transmitted per channel use, in reliable form.

In the classical channel coding, the $R$ channel rate of the classical channel $N$ can be defined by the $n$ number of channel uses, or the copies if the channel, and the $M$ bits of classical information can be sent through the classical channel $N$ faithfully, as follows:

$$R = \frac{1}{n} M . \tag{27}$$

The classical bits can be sent through reliable the classical channel, only if this rate does not exceed $C$, the *capacity* of the classical communication channel $N$, thus, if $R \le C$ holds. On the other hand, if the rate $R$ at which the classical information is transmitted over the classical channel exceeds the $C$ classical capacity of the classical channel $N$, i.e. $R > C$, the information cannot be transmitted through the channel in a reliable form. As follows, if $R > C$ holds, then the decoding probability of the sent information converges to zero in the number of channel uses.

The $C$ capacity of a noisy classical communication channel $N$, can be expressed by the maximum of the mutual information $I(A:B)$, which measures the amount of information between two random variables $A$ and $B$ as

$$I(A:B) = H(A) + H(B) - H(A,B), \tag{28}$$

where $H(A,B)$ is the joint entropy. The $C$ classical capacity of the classical communication channel $N$ takes its maximum over all possible input distributions $p(x)$ for the sender's sequence $A_n$, which consists of random variables, as follows:

$$C(N) = \max_{p(x)} I(A:B). \tag{29}$$

This $C(N)$ capacity describes the capacity of a classical communication channel, - as it can be concluded, it can be determined by a "single-use" formula, hence there is no any asymptotic nature in the case of a noisy classical communication channel. As we will see, this single-use nature does not hold anymore in the case of a quantum communication channel.

### B. Quantum Channel Coding

The noisy quantum channel encoding method uses very similar theoretical background as the classical channel coding, however there are some fundamental differences between them. The encoding and the decoding mathematically can be described by the superoperators $\mathcal{E}$ and $\mathcal{D}$, realized on the blocks of quantum states. The sender encodes the message from the source into a quantum state, and sends the encoded quantum state through the quantum channel. The receiver decodes the quantum state. The received quantum state will be a modified state (typically), according to the noise of the quantum channel.

The model of noisy quantum channel coding with the encoding and decoding process is illustrated in Fig. 7.

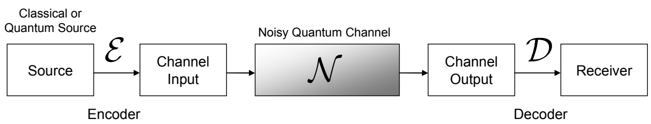


Fig. 7 Noisy Quantum Channel Coding

Similar to classical channel encoding, the quantum states can be transmitted over copies of a quantum channel. In this case, we have $n$ copies of a quantum channel $\mathcal{N}$, which will be denoted as $\mathcal{N}^{\otimes n}$. This channel model can be used to describe the transmission of codewords through the quantum channel, and it can be used to compute the asymptotic capacity of the quantum channel.

The information transmission over $n$ copies of quantum channel $\mathcal{N}$ is shown in Fig. 8. The input of the encoder consist of $m$ pure quantum states, the encoder maps the $m$ quantum states into the *joint state of n intermediate systems*. Each of these intermediate systems are sent through an independent instance of the quantum channel $\mathcal{N}$. This intermediate joint state is decoded by the $\mathcal{D}$ decoder, which results in $m$ quantum states. The output of the decoder $\mathcal{D}$ is typically a mixed quantum state, according to the noise of the quantum channel. The rate of the code is equal to the $m$ length of input codeword per $n$, the number of independent instances of the quantum channel.

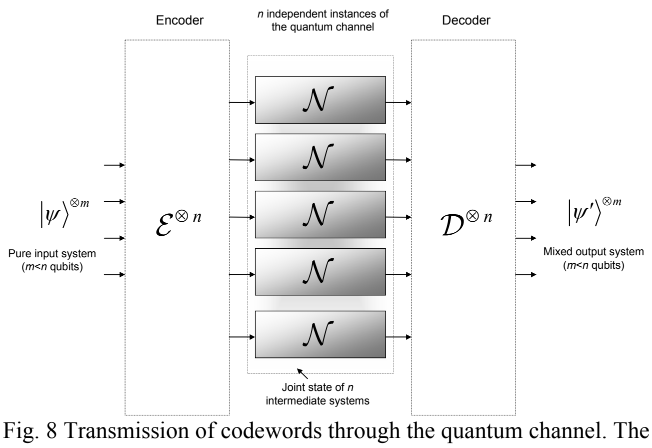


Fig. 8 Transmission of codewords through the quantum channel. The pure input quantum state consist of $m$ qubits, the encoder produces a joint state of $n$ intermediate systems. The encoded qubits are passed through the independent instances of the quantum channel

The encoding and decoding with $\mathcal{N}^{\otimes n}$ will have great relevance in the description of the asymptotic formula of the various channel capacities of the quantum channel.

In the case of quantum channel coding, using *n* copies of the quantum channel $\mathcal{N}$, the rate at which information can be transmitted through the quantum channel with arbitrarily small probability of error is

$$R = \frac{1}{n} \log M, \quad (30)$$

where $M$ denotes the set of possible codewords to be transmitted. The exponentially small probability of error at this rate can be achieved only if $R \leq C$, otherwise the probability of the successful decoding exponentially tends to zero, as the number of channel uses increases.

In the case of a classical communication channel *N*, the capacity is equal to the maximum of the mutual information between the sender and the receiver, for a *single-use* of the channel. It has an important conclusion: the asymptotic capacity of a classical communication channel *N*, is equal to the single use of the channel. On the other hand, it does not hold anymore in the case of quantum communication channels.

Our geometrical analysis is focused on the mixed quantum state, received by Bob. Alice's pure state is denoted by $\rho_A$, the noise is modeled by an affine map $\mathcal{N}$ and Bob's mixed input state is denoted by $\mathcal{N}(\rho_A) = \sigma_B$. For random variables $X$ and $Y$, $H(X,Y) = H(X) + H(Y|X)$, where *H(X)*, $H(X,Y)$ and $H(Y|X)$ are defined by probability distributions.

We measure in a geometrical representation the information which can be transmitted in the presence of noise on the quantum channel.

### C. The Asymptotic Classical Capacity of The Quantum Channel

This capacity is simply analogous to classical information transmission over a classical quantum channel. Hence, it simply gives the best rate at which a quantum channel $\mathcal{N}$ can be used to transmit classical information from Alice to Bob, which can be measured as the maximization of the mutual information between Alice and Bob:

$$C(\mathcal{N}) = \max_{A} I(A:B) = \max_{A} \left( H(A) - H(A|B) \right), \quad (31)$$

hence as the maximization of $I(A:B)$ over Alice's random input variables *A*, and Bob's variable *B*.

To measure the capacity which can be achieved by a quantum channel, we have to use an asymptotic formula. The asymptotic formula "activates" the benefits of quantum phenomena, such as entanglement and joint measurement.

The general sketch of classical capacity $C(\mathcal{N})$ of a quantum channel is illustrated in Fig. 9. Here we show the *asymptotic* classical channel capacity of the quantum channel. The asymptotic classical capacity of the quantum channel can be expressed as:

$$C(\mathcal{N}) = \lim_{n \to \infty} \frac{1}{n} \chi\left( \mathcal{N}(\rho_A)^{\otimes n} \right), \quad (32)$$

where $\rho_A$ is Alice's input system, *n* is the number of channel uses.

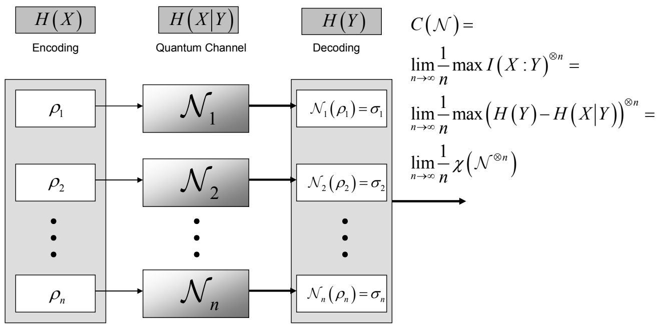


Fig. 9 The asymptotic classical capacity of a quantum channel. This capacity quantifies the maximum of the total number of transmittable classical bits divided by the number of channel uses

In the classical communication model, we seek to maximize $H(X)$ and minimize $H(X|Y)$ in order to maximize the radius of the smallest enclosing ball of Bob, since the radius can be computed as

$$r^* = \max_{\{all\ possible\ x_i\}} H(X) - H(X|Y). \quad (33)$$

To compute the radius $r^*$ of the smallest informational ball of quantum states and the entropies between mixed quantum states, instead of the classical Shannon entropy, we use the Holevo-Schumacher-Westmoreland (HSW) channel capacity [15], [16].

### D. Channel Entropies

According to the *HSW* theorem, the single use capacity $C^{(1)}(\mathcal{N})$ of a quantum channel $\mathcal{N}$, can be defined as follows [15], [16]:

$$\begin{aligned} C^{(1)}(\mathcal{N}) &= \max_{\{\text{all possible } p_i \text{ and } \rho_i\}} \mathcal{X}_{output} \\ &= \max_{p_1,\ldots,p_n,\rho_1,\ldots,\rho_n} \mathsf{S}\left( \mathcal{N}\left( \sum_{i=1}^{n} p_i(\rho_i) \right) \right) + \sum_{i=1}^{n} p_i \mathsf{S}\left( \mathcal{N}(\rho_i) \right), \end{aligned} \quad (34)$$

where $\mathcal{X}_{output}$ is the *Holevo quantity* of the output, $\mathsf{S}(\rho) = -Tr(\rho \log \rho)$ is the *von Neumann entropy*, and $\mathcal{N}(\rho_i)$ represents the output density matrix obtained from the quantum channel input density matrix $\rho_i$ [15]. Using the result of the HSW theorem [15], we will refer to the single use channel capacity as the radius of the smallest enclosing ball as follows:

$$r^* = C^{(1)}(\mathcal{N}) = \max_{\{\text{all possible } p_i \text{ and } \rho_i\}} \mathcal{X}_{output}. \quad (35)$$

In this paper, we use the geometrical interpretation of HSW channel capacity, using quantum relative entropy as a distance measure function.

## IV. ADVANCED PROPERTIES OF A QUANTUM CHANNEL

The additivity property and capacity recovery of zero-capacity quantum channels can be analyzed in a geometrical representation, which uses the quantum relative entropy-based channel capacity. There are some geometrical approaches in the literature, which could help to reveal the structural properties of quantum channel *additivity* [6], [7]. These methods also could be a very efficient tool for analyzing the still open questions on the additivity property of quantum channel capacity.

The quantum informational distance has some distance-like properties, however it is *not commutative* [16].

The classical communication over quantum channel is studied by Holevo, who showed the upper bound for quantum channel capacity [2], [14], [15], [16]. The capacity of quantum channel has been theoretically presented by Schumacher-Westmoreland, and they have proved that the upper bound of the quantum channel capacity can be attained [15], [16]. The capacity of quantum channel has been studied by Shor, who proved the equivalence of the additivity of the quantum channel capacity and the additivity of the minimum entropy output [2]. To compute the channel capacity of a quantum channel, instead of the classical Shannon entropy, we introduce the *Holevo-Schumacher-Westmoreland (HSW)* channel capacity [15], [16]. In the last decade, one of the most important questions in quantum information processing was the analysis of the impact of noise on the different channel capacities.

At present, the conjectures connected the quantum channel additivity are still not solved, some of them are only confirmed to hold for some classes of quantum channels. Currently, *the most basic questions on the classical capacity of a quantum channel still remain open*. The open questions related to the additivity of quantum channel capacities can be discussed by informational geometric approaches [6], [7], [9], [14].

### *A. Additivity of Quantum Channels*

To this day, additivity for quantum channel capacity $C(\mathcal{N}_1 \otimes \mathcal{N}_2)$ of two general quantum channels $\mathcal{N}_1$ and $\mathcal{N}_2$ has been *conjectured*, but still not proven. The equality of channel capacities is known for some special cases, but the generalized rule is still *unknown*. In classical systems, the correlations between classical inputs $\{x_1, x_2 \ldots x_m\}$, do not increase the capacity of the classical channel [2]. The classical channel behavior can be achieved in quantum communication, if the correlation between the input states is not allowed. In this case, the entangled input states are not allowed, and the joint quantum channel capacity $C(\mathcal{N}_1 \otimes \mathcal{N}_2)$ cannot be improved quantum mechanically. If the entanglement is allowed between the input states, then un-correlated input states $\{x_1, x_2 \ldots x_m\} \rightarrow |\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle$ can be encoded to entangled states as

$$\{x_1, x_2 \ldots x_m\} \rightarrow |\Psi_{12}\rangle \otimes |\Psi_{34}\rangle \otimes \ldots \otimes |\Psi_{(n-1)n}\rangle, \quad (36)$$

where $|\Psi_{i(i+1)}\rangle$ denotes the entangled states of the *i*-th and *i+1*-th density matrix inputs.

The entanglement between input states $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle$ is restricted to these $|\Psi_{i(i+1)}\rangle$ pairs, as we have illustrated it in Fig. 10.

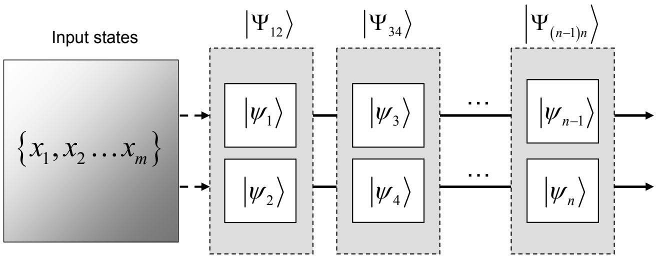


Fig. 10 Encoding with entanglement

For the additivity of any two quantum channels $\mathcal{N}_1$ and $\mathcal{N}_2$, it is *conjectured* that the following equation holds for the $C$ capacities:

$$C(\mathcal{N}_1 \otimes \mathcal{N}_2) = C(\mathcal{N}_1) + C(\mathcal{N}_2), \quad (37)$$

where $C(\mathcal{N}_1) + C(\mathcal{N}_2)$ is the total capacity if two channels $\mathcal{N}_1$ and $\mathcal{N}_2$ are used separately, while $C(\mathcal{N}_1 \otimes \mathcal{N}_2)$ is the joint capacity. For the additivity of quantum channels the following equation holds

$$C(\mathcal{N}_1 \otimes \mathcal{N}_2) = C(\mathcal{N}_1) + C(\mathcal{N}_2). \quad (38)$$

In Fig. 11, we illustrated the measurement setting for tensor product channel capacity $C(\mathcal{N}_1 \otimes \mathcal{N}_2)$. The two quantum channels $\mathcal{N}_1$ and $\mathcal{N}_2$ form a tensor product channel $\mathcal{N}_1 \otimes \mathcal{N}_2$, which is denoted by the dashed frame. The parallel quantum channels $\mathcal{N}_1$ and $\mathcal{N}_2$ can be viewed one channel denoted by $\mathcal{N}_{12}$, for which $\mathcal{N}_{12} : \mathcal{H}_{12} \rightarrow \mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}$ is the Hilbert-space. The additivity conjecture of quantum channel capacity can be stated as, how could the entangled states contribute to the capacity $C(\mathcal{N}_1 \otimes \mathcal{N}_2)$ of a product channel $\mathcal{N}_{12} = \mathcal{N}_1 \otimes \mathcal{N}_2$, or how could we exploit entanglement in quantum channel capacity [14]. The joint capacity $C(\mathcal{N}_1 \otimes \mathcal{N}_2)$ can be determined after a *joint measurement* of channels $\mathcal{N}_1$ and $\mathcal{N}_2$. The two quantum channels $\mathcal{N}_1$ and $\mathcal{N}_2$ form a tensor product channel $\mathcal{N}_{12} = \mathcal{N}_1 \otimes \mathcal{N}_2$.

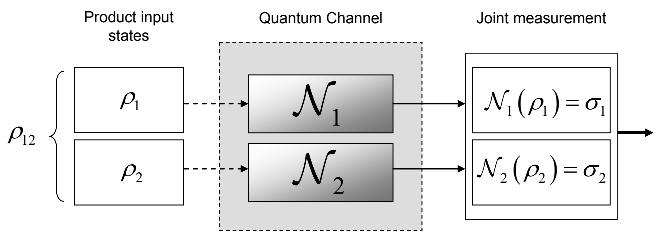


Fig. 11 Joint-measurement setting for tensor-product channel capacities

As it has been proven [14], if there is no joint measurement in the model, then the joint capacity $C(\mathcal{N}_1 \otimes \mathcal{N}_2)$ of quantum channels $\mathcal{N}_1$ and $\mathcal{N}_2$ is additive, and super-additivity property cannot be exploited:

$$C(\mathcal{N}_1 \otimes \mathcal{N}_2) = C(\mathcal{N}_1) + C(\mathcal{N}_2). \quad (39)$$

If for the joint capacity of the two channels $\mathcal{N}_1$ and $\mathcal{N}_2$ , the conjecture

$$C(\mathcal{N}_1 \otimes \mathcal{N}_2) > C(\mathcal{N}_1) + C(\mathcal{N}_2), \quad (40)$$

holds, then the tensor-product channel model $\mathcal{N}_{12} = \mathcal{N}_1 \otimes \mathcal{N}_2$ forms a "*superchannel*", for which *super-additivity* holds. If this property holds, then using entanglement on the input of channel $\mathcal{N}_{12} = \mathcal{N}_1 \otimes \mathcal{N}_2$, the channel capacity $C(\mathcal{N}_1 \otimes \mathcal{N}_2)$ can be increased, and the additivity property of channel capacity fails:

$$C(\mathcal{N}_1 \otimes \mathcal{N}_2) \neq C(\mathcal{N}_1) + C(\mathcal{N}_2). \quad (41)$$

We note, that the parallel quantum channel view $\mathcal{N}_{12} = \mathcal{N}_1 \otimes \mathcal{N}_2$, and the model of serial entangled inputs are equivalent from the view of super-additivity.

### B. *Capacity recovery of Quantum Channels*

The capacity recovery of the zero-error capacity of quantum channels makes it possible to use two quantum channels, each with zero zero-error capacity, with a positive joint zero-error capacity. The capacity recovery of quantum channels may be the starting-point of a large-scale revolution in quantum information theory and in the communication of future quantum networks [12], [13]. The *capacity recovery* of zero-capacity quantum channels makes it possible to use two zero-capacity quantum channels with a positive joint capacity at the output. The problem of capacity recovery can be discussed as part of a larger problem set – the problem of quantum channel additivity. The number of efficient approximation algorithms for quantum informational distances is very small, because of the special properties of quantum informational generator functions and of asymmetric quantum informational distances. If we wish to analyze the properties of quantum channels using today's classical computer architectures, an extremely efficient algorithm is needed [35], [36], [37].

With the help of efficient computational geometric methods, the *capacity recovery of zero-capacity quantum channels* can be analyzed very efficiently. Computational Geometry was originally focused on the construction of efficient algorithms and provides a very valuable and efficient tool for computing hard tasks. In many cases, traditional linear programming methods are not very efficient.

To analyze a quantum channel for a *large number* of input quantum states with classical computer architectures, very fast and efficient algorithms are required. We use these classical computational geometric tools to discover the still unknown "*superactive*" zero-capacity quantum channels [1].

The problem of capacity recovery of zero-capacity quantum channels can be viewed as a smaller subset of a larger problem set involving the additivity of quantum channels. The problem of capacity recovery of zero-capacity quantum channels can be viewed as a smaller subset of a larger problem set involving the additivity of quantum channels.

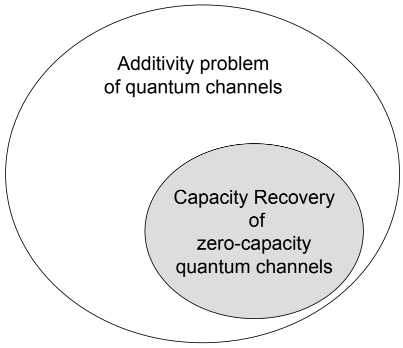


Fig. 12 The problem of capacity recovery of zero-capacity quantum channels as a sub domain of a larger problem set

The possibility of the capacity recovery of the zero capacity quantum channels can be a very valuable tool for improving the results of fault-tolerant quantum computation and possible communication techniques over noisy quantum channels in future's quantum networks.

The zero-error capacity of the quantum channel measures the amount of information which can be transmitted through a noisy quantum channel with a *zero probability of error* [23], [24], [25].

Since the revolutionary properties of capacity recovery of quantum channel capacities have been reported on, many new quantum informational results have been developed [1], [18], [19], [20], [21], [22]. The capacity recovery of zero-error capacity implies the fact that a possible combination of quantum channels with zero zero-error capacity exists, where individually totally useless channels can activate each other, and their joint zero-error capacity will be greater than zero [1].

## V. Geometrical Interpretation of Quantum Channel Capacity

The authors of [15] have shown that the capacity of a quantum channel can be measured geometrically, using quantum relative entropy function as a distance measure. Schumacher and Westmoreland have shown that the channel capacity of every optimal output state $\rho_k$ can be expressed as [15]

$$C^{(1)}(\mathcal{N}) = D(\rho_k \| \sigma), \quad (42)$$

where $\sigma = \sum p_k \rho_k$ is the optimal average output state and the relative entropy function of two density matrices can be defined as

$$D(\rho_k \| \sigma) = Tr\left[\rho_k \log(\rho_k) - \rho_k \log(\sigma)\right]. \quad (43)$$

In this definition, *Tr* is the trace operator. In conclusion, for non-optimal output states $\delta$ and optimal average output state $\sigma = \sum p_k \rho_k$, we have $C^{(1)}(\mathcal{N}) = D(\delta \| \sigma) \leq D(\rho_k \| \sigma)$. Moreover, in [15], Schumacher and Westmoreland have also shown that there exists at least one optimal output state $\{p_k, \rho_k\}$ which achieves the optimal capacity $C^{(1)}(\mathcal{N}) = D(\rho_k \| \sigma)$. The geometrical interpretation of quantum channel capacity was introduced in [15], using the quantum relative entropy function as a distance measure as follows:

$$C^{(1)}(\mathcal{N}) = r^* = \min_{\{\sigma\}} \max_{\{\rho\}} D(\mathcal{N}(\rho) \| \mathcal{N}(\sigma)). \quad (44)$$

We analyze the capacity recovery of the quantum channel by clustering and convex hull calculations based on quantum relative entropy. If we denote the optimal output states by $\{\mathcal{N}(\psi_k) = p_k, \rho_k\}$ which achieve the capacity $C^{(1)}(\mathcal{N})$ of channel $\mathcal{N}$ and $\sigma = \sum_k p_k \rho_k$ , then the single use quantum channel capacity can be derived in terms of the quantum relative entropy in the following way [16], [17]:

$$\begin{aligned}
&\sum_k p_k D(\rho_k \| \sigma) \\
&= \sum_k \left( p_k Tr[\rho_k \log(\rho_k)] - p_k Tr[\rho_k \log(\sigma)] \right) \\
&= \sum_k \left( p_k Tr[\rho_k \log(\rho_k)] \right) - Tr\left[ \sum_k \left( p_k \rho_k \log(\sigma) \right) \right] \quad (45) \\
&= \sum_k \left( p_k Tr[\rho_k \log(\rho_k)] \right) - Tr[\sigma \log(\sigma)] \\
&= \mathrm{S}(\sigma) - \sum_k p_k \mathrm{S}(\rho_k) = \mathcal{X}.
\end{aligned}$$

It can therefore be concluded that the *Holevo quantity* $\mathcal{X}$ can be expressed in terms of the quantum relative entropy and the $C^{(1)}(\mathcal{N})$ single use HSW channel capacity as [15]

$$C^{(1)}(\mathcal{N}) = \max_{\{all\ p_k, \psi_k\}} \sum_k p_k D(\mathcal{N}(\psi_k) \| \mathcal{N}(\psi)), \quad (46)$$

where $\psi_k$ denotes the *input* quantum states of channel $\mathcal{N}$ and $\psi = \sum_k p_k \psi_k$. The geometric interpretation of the HSW channel capacity has been studied by Cortese [14], who also extended these results to the general qudit channels.

Using the resulting quantum relative entropy function [15] and the HSW-theorem, the $C(\mathcal{N})$ *asymptotic* classical capacity of the quantum channel can be expressed with the help of the radii of the smallest quantum informational balls as follows:

$$\begin{aligned}
&r^*_{super}(\mathcal{N}) = C(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} C^{(1)}(\mathcal{N}^{\otimes n}) = \\
&\lim_{n\to\infty} \frac{1}{n} \left( \sum_{i=1}^{n} r_i^* \right) = \lim_{n\to\infty} \frac{1}{n} \max_{p_1,\ldots,p_n,\rho_1,\ldots,\rho_n} (\mathcal{X}_{AB})^{\otimes n},
\end{aligned} \quad (47)$$

where $r_i^*$ is the single use capacity of the $i$-th use of quantum channel $\mathcal{N}$ , $\rho_k^{AB}$ is the optimal output channel state, and $\sigma^{AB}$ is the average state. We analyze the capacity recovery property of the quantum channel, using the *mini-max* criterion for states $\rho_k^{AB}$ and $\sigma^{AB}$. The radius $r^*_{super}$ of the superball is equal to the asymptotic classical capacity [6], [11].

In Fig. 13, we illustrate the superball representation for the analysis of two quantum channels, however it naturally can be extended to $n$ different quantum channel models. The geometrical structure of quantum informational balls differs from the geometrical structure of ordinary Euclidean balls.

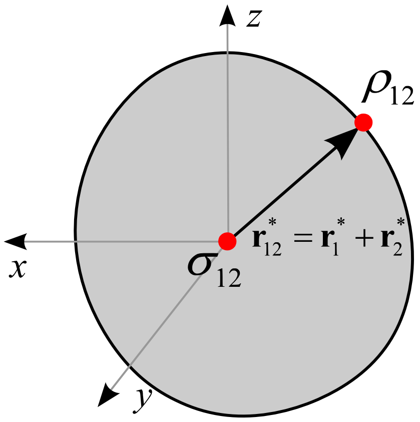


Fig. 13 Geometric interpretation of capacity recovery of very noisy quantum channels

In the capacity recovery problem, we have to use *different* quantum channel models [1]. For two different quantum channels $\mathcal{N}_1^{\otimes n}$ and $\mathcal{N}_2^{\otimes n}$, the *asymptotic* HSW channel capacity $C(\mathcal{N}_1 \otimes \mathcal{N}_2)$ is equal to the sum of the radii $r^*_{super}(\mathcal{N}_1)$ and $r^*_{super}(\mathcal{N}_2)$ of the quantum informational superballs, whose radii form a new quantum superball with radius

$$\begin{aligned}
&r^*_{super}(\mathcal{N}_1 \otimes \mathcal{N}_2) = C(\mathcal{N}_1 \otimes \mathcal{N}_2) = \\
&\lim_{n\to\infty} \frac{1}{n} C^{(1)}\left( (\mathcal{N}_1 \otimes \mathcal{N}_2)^{\otimes n} \right) = r^*_{super}(\mathcal{N}_1) + r^*_{super}(\mathcal{N}_2).
\end{aligned} \quad (48)$$

In Fig. 14 we show the measurement setting to analyze the capacity recovery of very noisy optical quantum channel-pair $\mathcal{N}_1$ and $\mathcal{N}_2$ .

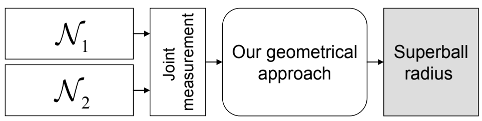


Fig. 14 The method of capacity recovery of optical quantum channels

The geometrical method computes the joint capacity, based on the clustering of channel output quantum states and a convex hull calculation [12], [13], [14].

## *A. The Quantum Informational Ball*

We use the Delaunay tessellation, since it is the *fastest known tool* to seek the center of the smallest enclosing ball of points. The circumcircle of the given quantum states is the circle that passes through the quantum states $\rho_1$ and $\rho_2$ of the edge $\rho_1\rho_2$ and endpoints $\rho_1$, $\rho_2$ and $\rho_3$ of the triangle $\rho_1\rho_2\rho_3$. The triangle $t$ is said to be *Delaunay*, when its *circumcircle* is *empty* [3], [7]. For an empty circumcircle, the circle passing through the quantum states of a triangle $t \in T$ and encloses no other vertex of the set $\mathcal{S}$ . The quantum Delaunay diagrams between mixed quantum states are different from Euclidean diagrams, as we have illustrated Fig. 15.

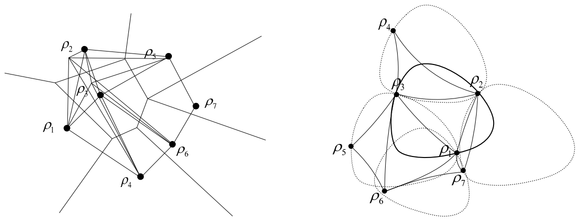


Fig. 15 Classical Euclidean (a), and quantum Delaunay tessellation (b)

The problem of clustering in quantum space, using the quantum informational distance as a distance function, is a completely new area in quantum information theory. The properties of Voronoi diagrams [3] in quantum space have been studied by Kato *et al*. [10], however the problem of clustering was not analyzed in their work. The coreset method for different distances has been studied in the literature [4], [7], [9].

## VI. The Optical Quantum Channel

The optical quantum channel $\mathcal{N}$ can be described in the Kraus representation [2], [5], using a set of Kraus matrices $\mathcal{A}=\{A_i\}$, in the following form

$$\mathcal{N}(\rho)=\sum_i A_i \rho A_i^{\dagger}, \tag{49}$$

where $\sum_i A_i^{\dagger} \rho A_i = I$, and

$$A_1=\begin{bmatrix} \sqrt{p} & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } A_2=\begin{bmatrix} 0 & 0 \\ \sqrt{1-p} & 0 \end{bmatrix}, \tag{50}$$

where $p$ represents the probability that the channel leaves the $|0\rangle$ input state unchanged. In practical applications, this parameter represents the probability of energy loss from losing a particle. The channel flips the input state from $|0\rangle$ to $|1\rangle$ with probability $1-p$. In the Bloch sphere representation, the effect of the amplitude damping channel on the initial input state $\rho=\frac{1}{2}\left(\mathbf{1}+|\mathbf{r}_{in}|\right)$, where $|\mathbf{r}_{in}|$ is the length of the initial Bloch vector, can be analyzed. The output state is denoted by $\mathcal{N}(\rho)=\frac{1}{2}\left(\mathbf{1}+|\mathbf{r}_{out}|\right)$, hence the amplitude damping channel can be expressed using Bloch vectors $\mathbf{r}_{in}$ and $\mathbf{r}_{out}$ in the following way:

$$\mathbf{r}_{out}=\begin{pmatrix} \mathbf{r}_{out}^{(x)} \\ \mathbf{r}_{out}^{(y)} \\ \mathbf{r}_{out}^{(z)} \end{pmatrix}=\begin{pmatrix} \sqrt{1-p} & 0 & 0 \\ 0 & \sqrt{1-p} & 0 \\ 0 & 0 & 1-\frac{p}{2} \end{pmatrix}\begin{pmatrix} \mathbf{r}_{in}^{(x)} \\ \mathbf{r}_{in}^{(y)} \\ \mathbf{r}_{in}^{(z)} \end{pmatrix}+\begin{pmatrix} 0 \\ 0 \\ \frac{p}{2} \end{pmatrix}. \tag{51}$$

The smallest value of $D(\rho\|\sigma)$ corresponds to the contour closest to the location of the density matrix. In Fig. 16(a), the Euclidean distances from the origin of the Bloch sphere to center $\mathbf{c}^*$ and to point $\rho$ are denoted by $m_\sigma$ and $m_\rho$, respectively. To determine the optimal length of vector $\mathbf{r}_\sigma$, the algorithm moves point $\sigma$. As we move vector $\mathbf{r}_\sigma$ from the optimum position, the larger contour corresponding to a larger value of quantum relative entropy $D$ will intersect the channel ellipsoid surface, thereby increasing $\max_{\mathbf{r}_\rho} D(\mathbf{r}_\rho\|\mathbf{r}_\sigma)$. The optimal quantum informational ball is illustrated in light-grey in Fig. 16(b).

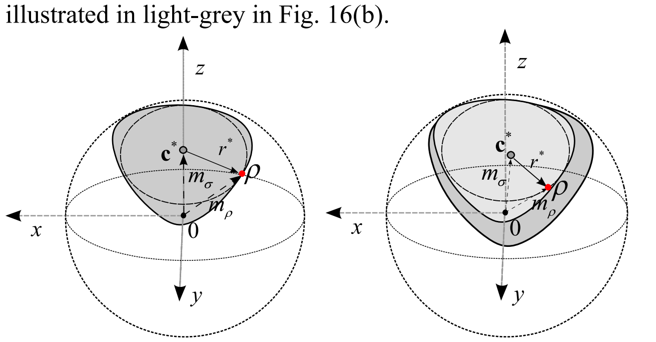


Fig. 16 Intersection of quantum informational ball and channel ellipsoid of amplitude damping channel

From our geometrical analysis, it can be concluded that the optimum input states for an optical quantum channel are unentangled, non-orthogonal quantum states [17]. In Fig. 17, we show the results for the capacity recovery analysis of amplitude damping channel. The capacity recovery of amplitude damping channel $\mathcal{N}$ cannot be described by the relation derived for unital quantum channels. The radii of the smallest quantum informational balls of channels $\mathcal{N}_1$ and $\mathcal{N}_2$ are denoted by $\mathbf{r}_1^*$ and $\mathbf{r}_2^*$.

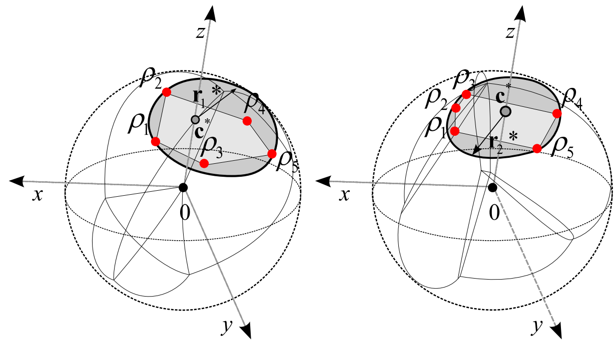


Fig. 17 Smallest quantum informational balls of two independent quantum channels

Our analysis has shown that the optimum input states are non-orthogonal input quantum states [16].

### A. *Analysis of Channel Output States*

A coreset of a set of output quantum states has the same behavior as the larger input set, so clustering and other approximations can be made with smaller coresets. The coreset can be viewed as a smaller input set of channel output states, hence it can be used as the input to an approximation algorithm. The weighted sum of errors of the smaller coreset is a $(1\pm\varepsilon)$ approximation of the larger input set. These coresets are called weak coresets [9] and this method can be applied in quantum space between quantum states. The weak coresets include all the relevant information required to analyze the original extremely large input set. The coreset approach has significantly lower computational complexity, hence it can be applied very efficiently in the quantum space [9].

### B. Clustering Channel Output States

Using $\mu$-similar quantum informational distances and the $\mathcal{W}$-weak coreset of quantum states, the capacity recovery of very noisy optical quantum channels can be analyzed by an $(1+\varepsilon)$-approximation algorithm in a run time

$$\mathcal{O}\left(d^2 2^{\frac{k}{\varepsilon}} \log^{k+2} n + dkn\right), \tag{52}$$

where $k$ is the number of quantum states in set $\mathcal{S}_{OUT}$, $n$ is the number of input states and $d$ is the dimension of the points.
To summarize, our algorithmic capacity recovery of very noisy optical quantum channels combines the weak coreset methods and the clustering algorithms.

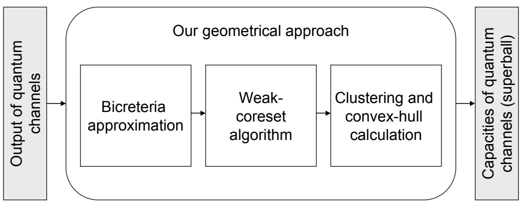


Fig. 18 Decomposition of our geometrical approach for finding recoverable very noisy optical channels

In Fig. 19, we illustrate the clustering method of channel output states. In the clustering process, our algorithm computes the median-quantum states denoted by $\sigma_i$, using a fast weak coreset and clustering algorithm. In the next step, we compute the convex hull of the median quantum states and, from the convex hull, the radius of the smallest quantum informational ball can be obtained. The convex-hull calculation is based on the quantum Delaunay diagrams. The smallest superball measures the channel capacity; hence the radius of the superball is equal to the sum of radii of quantum balls of independent channel outputs. The output states are measured by a joint measurement setting.

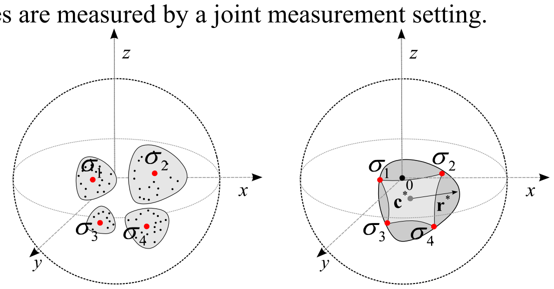

Fig. 19 Capacity of quantum channel analyzed by the radius of the smallest quantum informational ball

Using the modified weak coreset algorithm and the $(1+\varepsilon)$-approximation algorithm, the capacity recovery of quantum channels can be analyzed relative to $\mu$-similar quantum informational distances and $k$ median-quantum states with error $error(\mathcal{S}_{IN}, \mathcal{S}_{OUT}) \leq (1+7\varepsilon)\, opt_k(\mathcal{S}_{IN})$, where $opt_k(\mathcal{S}_{IN})$ is the error of the optimal solution for set of input quantum states $\mathcal{S}_{IN}$.

## VII. Conclusion

This paper shows a fundamentally new algorithmic solution for capacity recovery of very noisy optical quantum channels. To analyze channel capacity recovery, we introduced the quantum informational "*superball*" representation. The iterations are based on the computed radius of the superball. The algorithm presented has lower complexity in comparison with other existing coreset and approximation algorithms, which can also be applied in quantum space. This paper is intended to be an introduction to the basic properties of the proposed framework for finding recoverable very noisy optical quantum channels. The proposed method can be a very valuable tool for improving the results of fault-tolerant quantum computation and possible communication techniques over noisy optical quantum channels. In future work, we would like to extend our results to other possible very noisy quantum channel models and we would like to show some typical results on recovered very noisy quantum channels.

## Acknowledgment

The authors would like to thank the financial support from „*Sandor Csibi*" Ph.D. Researcher Scholarship at the Budapest University of Technology, Faculty of Electrical Engineering and Informatics, Hungary.

## References

[1] G. Smith, J. Yard, Quantum communication with very noisy channels. *Science 321*, 1812-1815 (2008)

[2] S. Imre, F. Balázs: *Quantum Computing and Communications – An Engineering Approach*, Published by John Wiley and Sons Ltd, 2005.

[3] F. Aurenhammer and R. Klein. Voronoi Diagrams. In J. Sack and G. Urrutia (Eds), *Handbook of Computational Geometry*, Chapter V, pp. 201–290. Elsevier Science Publishing, 2000. 03

[4] M. Badoiu, S. Har-Peled, and P. Indyk. Approximate clustering via core-sets. In *Proceedings 34th ACM Symposium on Theory of Computing*, pages 250–257, 2002.

[5] F. Nielsen, R. Nock: On the smallest enclosing information disk. Inf. Process. Lett. IPL'08, 105(3): 93-97 (2008)

[6] L. Gyongyosi, Sandor Imre: Novel Geometrical Solution to Additivity Problem of Classical Quantum Channel Capacity, In *Proceedings of The 33rd IEEE Sarnoff Symposium* - 2010, Princeton University, Apr. 2010, Princeton, New Jersey, USA.

[7] M. R. Ackermann, J.s Blömer, and C. Sohler. Clustering for metric and non-metric distance measures. In *Proceedings of the 19th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '08)*, pages 799–808., 2008.

[8] W. J. Munro, K. A. Harrison, A. M. Stephens, S. J. Devitt, Kae Nemoto, From quantum fusiliers to high-performance networks arXiv:0910.4038 (2009)

[9] F. Nielsen, J. D. Boissonnat, and R. Nock. On Bregman Voronoi diagrams. In Proceedings of the 18th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'07), pages 746–755, Philadelphia, PA, USA, 2007. Society for Industrial and Applied Mathematics.

[10] K. Kato, M. Oto, H. Imai, and K. Imai, "Voronoi diagrams for pure 1-qubit quantum states,quant-ph/0604101, 2006.

[11] L. Gyongyosi, S. Imre, Computational Geometric Analysis of Physically Allowed Quantum Cloning Transformations for Quantum Cryptography, In *Proceedings of the 4th WSEAS International Conference on Computer*

*Engineering and Applications (CEA '10).* Harvard University, Cambridge, USA. 2010. pp. 121-126.

[12] L. Gyongyosi, S. Imre, Computational Information Geometric Analysis of Quantum Channel Additivity, *Photon10 Conference*, Quantum Electronics Group (QEP-19), 2010, Southampton, UK.

[13] L. Gyongyosi, S. Imre, Information Geometrical Analysis of Additivity of Optical Quantum Channels, *IEEE/OSA Journal of Optical Communications and Networking (JOCN)*, IEEE Photonics Society & Optical Society of America, ISSN: 1943-0620; 2010.

[14] J. A. Cortese, "The Holevo-Schumacher-Westmoreland Channel Capacity for a Class of Qudit Unital Channels", 2002, LANL ArXiV e-print quant-ph/0211093.

[15] B. W. Schumacher and M. Westmoreland, "Relative Entropy in Quantum Information Theory" 2000, LANL ArXiV e-print quant-ph/0004045.

[16] B. W. Schumacher and M. Westmoreland, "Optimal Signal Ensembles", 1999, LANL ArXiV e-print quant-ph/9912122.

[17] M. Asano, M. Ohya, Quantum, Teleportation with Non-Maximal Entangled State, *Proceedings of the 14th WSEAS International Conference on Applied Mathematics (MATH '09),* 2009. pp. 191-195.

[18] R. Duan, Superactivation of zero-error capacity of noisy quantum channels.arXiv:0906.2527 [quant-ph], 2009.

[19] T. Cubitt, J. X. Chen, and A. Harrow, arXiv: 0906.2547.

[20] R. Duan, Y. Feng, Z. F. Ji, and M. S. Ying, *Phys. Rev. Lett. 98*, 230502 (2007).

[21] T. Cubitt, A. Montanaro, and A. Winter, arXiv: 0706.0705 [quant-ph] (2007).

[22] T. Cubitt, A. Harrow, D. Leung, A. Montanaro, and A. Winter, Comm. *Math. Phys. 284(*1), 281 (2008).

[23] R. C. Medeiros and F. M. de Assis, *Int. J. Quant. Inf. 3*, 135 (2005);

[24] R. C. Medeiros, R. Alleaume, G. Cohen, and F. M. de Assis, quant-ph/0611042.

[25] S. Beigi and P. W. Shor, arXiv:0709.2090 [quant-ph] (2007).

[26] J. A. Cortese, "The Holevo-Schumacher-Westmoreland Channel Capacity for a Class of Qudit Unital Channels", 2002, LANL ArXiV e-print quant-ph/0211093.

[27] R. Nock, F. Nielsen: Fitting the Smallest Enclosing Bregman Ball. ECML 2005, ECML'05, pages 649-656.

[28] F. Nielsen, R. Nock: On the smallest enclosing information disk. Inf. Process. Lett. IPL'08, 105(3): 93-97 (2008)

[29] F. Nielsen, R. Nock: Quantum Voronoi Diagrams and Holevo Channel Capacity for 1-Qubit Quantum States, ISIT 2008.

[30] F. Nielsen, R. Nock: Bregman Sided and Symmetrized Centroids. ICPR 2008, ICPR'08, (arXiv:0711.3242)

[31] F. Nielsen, R. Nock: Approximating Smallest Enclosing Balls with Application to Machine Learning, *International Journal on Computational Geometry and Applications (IJCGA'09)*, 2009.

[32] M. M.Wolf and J. Eisert, Classical information capacity of a class of quantum channels. *New Journal of Physics*, 7(93), 2005.

[33] M. Hayashi and H. Nagaoka, "General formulas for capacity of classical-quantum channels," *IEEE Transactions on Information Theory,* Vol.49, No.7, pp.1753-1768 (2003).

[34] M. Hayashi, H. Imai, K. Matsumoto, M. B. Ruskai and T. Shimono, "Qubit channels which require four inputs to achieve capacity: Implications for additivity conjectures," *Quantum Information and Computation, 5, 032-040 (2005).*

[35] L. Gyongyosi, S. Imre, Computational Geometric Analysis of Physically Allowed Quantum Cloning Transformations for Quantum Cryptography, In *Proceedings of the 4th WSEAS International Conference on Computer Engineering and Applications (CEA '10).* Harvard University, Cambridge, USA. 2010. pp. 121-126.

[36] L. Gyongyosi, S. Imre, Information Geometrical Solution to Additivity of Non-Unital Quantum Channels, QCMC 2010, *10th Quantum Communication, Measurement & Computing Conference*, July 2010, University of Queensland, Brisbane, Queensland, Australia.

[37] L. Gyongyosi, S. Imre, Capacity Recovery of Useless Photonic Quantum Communication Channels, *ALS Conference*, Lawrence Berkeley National Laboratory (Berkeley Lab), University of California, Berkeley (California), USA, 2010.

[38] Satoshi Iriyama, Masanori Ohya, Mathematical Characterization of Quantum Algorithm, *Proceedings of the 14th WSEAS International Conference on Applied Mathematics (MATH '09)*, 2009. pp. 184-190.

[39] Masanari Asano, Masanori Ohya, Quantum, Teleportation with Non-Maximal Entangled State, *Proceedings of the 14th WSEAS International Conference on Applied Mathematics (MATH '09),* 2009. pp. 191-195.

[40] Yuji Hirota, A Categorical Approach to Quantum Algorithm, Proceedings of the 14th WSEAS International Conference on Applied Mathematics (MATH '09), 2009. pp. 268-270.

[41] Miroslav Svitek, Physics of Information Representation, Transmission and Processing, Recent Advances On Data Networks, Communications, Computers, *Proceedings of the 8th WSEAS International Conference on Data Networks, Communications, Computers, (DNCOCO '09),* 2009. pp. 205-212

[42] Mark Burgin, Mathematical Theory of Information Technology, Proceedings of the 8th WSEAS International Conference on Data Networks, Communications, Computers, (DNCOCO '09), 2009. pp. 42-48

**Laszlo Gyongyosi** received the M.Sc. degree in Computer Science with Honors from the Budapest University of Technology and Economics (BUTE) in 2008. He is a final year Ph.D. Candidate at the Department of Telecommunications, BUTE. His research interests are in Quantum Computation and Communication, Quantum Channel Capacity, Quantum Cryptography and Quantum Information Theory. He is currently completing a book on advanced quantum communications, and he teaches courses in Quantum Computation. In 2009, he received Future Computing Best Paper Award on quantum information, in 2010, he was awarded the Best Paper Prize of University of Harvard, USA. In 2010, he obtained a Ph.D. Researcher Grant from University of Arizona, USA.

**Sandor Imre** was born in Budapest in 1969. He received the M.Sc. degree in Electronic Engineering from the Budapest University of Technology (BUTE) in 1993. Next he started his Ph.D. studies at BUTE and obtained dr. univ. degree in 1996, Ph.D. degree in 1999 and DSc degree in 2007. Currently he is carrying his teaching activities as Head of the Dept. of Telecommunications of BUTE. He was invited to join the Mobile Innovation Centre of BUTE as R&D director in 2005. His research interest includes mobile and wireless systems, quantum computing and communications. Especially he has contributions on different wireless access technologies, mobility protocols and reconfigurable systems.