Cones of Critical Directions in Optimal Control

Javier F. Rosenblueth* and Gerardo Sánchez Licea†

Abstract—In this paper we study a fundamental aspect of the theory of second order necessary conditions for certain classes of optimal control problems involving equality and/or inequality constraints in the control. It is well-known that, under certain normality assumptions, a certain quadratic form is nonnegative on a cone of critical directions (or differentially admissible variations). The purpose of this paper is to characterize normality in terms of some regularity assumptions and to illustrate through several examples the fact that, by assuming those types of regularity, the result may fail to hold on the former and larger sets of critical directions.

Keywords—Optimal control, second order conditions, equality and/or inequality constraints, normality

I. Introduction

This paper deals with second order necessary conditions for certain classes of optimal control problems posed over piecewise continuous controls and involving equality and/or inequality constraints in the control functions. The importance of deriving such conditions from a theoretical point of view as well as due to a wide range of applications is fully explained in [1, 2, 5, 8–13, 15–21] and references therein.

There is an extensive literature on second order conditions for optimal control problems and how the theory can be applied to practical problems. Different applications can be found in [11, 12] and, in particular, two problems posed in [12] can be studied by applying the theory that follows. One is the classical problem of a planar Earth-Mars orbit transfer with minimal transfer time, while the second deals with the Rayleigh problem with control constraints, that is, the control of current in a tunnel-diode oscillator. We refer to [12] for a full discussion of the two applications and how a solution is derived by solving numerically a Riccati equation. The question on the nonnegativity of a quadratic form posed in this paper plays a fundamental role in testing for possible candidates as optimal controls for such problems.

Before we state the problem we shall deal with, and for comparison reasons, let us first give a brief overview of well-known second order necessary conditions for other optimization problems which also involve equality and/or inequality constraints. We shall explicitly state, for those problems, two different conditions of second order which can be found in the literature and for which one implies the other, being the former in this sense a stronger condition than the latter.

A. The finite dimensional case

The approach we follow for the finite dimensional case, which yields well-known first and second order necessary conditions, is based on the notions of regularity and normality. A full account of these ideas can be found in [7].

Let us begin with a problem involving nonlinear equality constraints. Suppose we are given functions $f, g_1, \ldots, g_m$ mapping $\mathbf{R}^n$ to $\mathbf{R}$ ($m < n$) and we are interested in minimizing f on S where

$$S = \{x \in \mathbf{R}^n \mid g_\alpha(x) = 0 \ (\alpha \in A)\}$$

and $A = \{1, \ldots, m\}$. It will be assumed that the functions f, g_α ($\alpha \in A$) are of class C^2 on S.

For all $x_0 \in S$ define the set of *curvilinear tangent vectors of S at x_0* as

$$C_S(x_0) := \{h \in \mathbf{R}^n \mid \text{there exist } \epsilon > 0 \text{ and}$$

$$x\colon (-\epsilon, \epsilon) \to S \text{ such that } x(0) = x_0 \text{ and } \dot{x}(0) = h\}$$

and the set of vectors satisfying the *tangential constraints of S at x_0* as

$$R_S(x_0) := \{h \in \mathbf{R}^n \mid g'_\alpha(x_0)h = 0 \ (\alpha \in A)\}.$$

As one readily verifies, $C_S(x_0) \subset R_S(x_0)$ for all $x_0 \in S$, but the converse may not hold. If $C_S(x_0) = R_S(x_0)$, $x_0 \in S$ is said to be a *regular point of S*. This notion yields first and second order conditions

*IIMAS–UNAM, National Autonomous University of Mexico, Apartado Postal 20-726, Mexico DF 01000, Mexico (e-mail: jfrl@unam.mx).

†Faculty of Sciences, National Autonomous University of Mexico, Av Universidad 3000, Mexico DF 04510, Mexico (e-mail: gesl@ciencias.unam.mx).

in terms of the standard Lagrangian defined, for all $((x, \lambda) \in \mathbf{R}^n \times \mathbf{R}^m)$, by

$$F(x, \lambda) := f(x) + \sum_1^m \lambda_\alpha g_\alpha(x).$$

Theorem 1.1 *Suppose x_0 affords a local minimum to f on S. If x_0 is a regular point of S then there exists $\lambda \in \mathbf{R}^m$ such that $F_x(x_0, \lambda) = 0$ and $\langle h, F_{xx}(x_0, \lambda)h \rangle \geq 0$ for all $h \in R_S(x_0)$.*

In general it is difficult to test for regularity and one criterion, easier to verify, is that of normality. We shall say that $x_0 \in S$ is a *normal point of* S if the linear equations

$$g'_\alpha(x_0; h) := g'_\alpha(x_0)h = 0 \quad (\alpha \in A)$$

in h are linearly independent, that is, if the gradients $g'_1(x_0), \ldots, g'_m(x_0)$ are linearly independent, which is equivalent to the requirement that the matrix

$$\left(\frac{\partial g_\alpha(x_0)}{\partial x^i} \right) \qquad (\alpha = 1, \ldots, m;\ i = 1, \ldots, n)$$

be of rank m. It is well-known that, if x_0 is a normal point of S then it is a regular point of S and, moreover, the multiplier λ in Theorem 1.1 is unique. This yields the following result.

Theorem 1.2 *Suppose x_0 affords a local minimum to f on S. If x_0 is a normal point of S then there exists a unique $\lambda \in \mathbf{R}^m$ such that $F_x(x_0, \lambda) = 0$. Moreover, $\langle h, F_{xx}(x_0, \lambda)h \rangle \geq 0$ for all $h \in R_S(x_0)$.*

Let us now change the data of the problem and suppose that

$$S = \{x \in \mathbf{R}^n \mid g_\alpha(x) \leq 0\ (\alpha \in A),$$
$$g_\beta(x) = 0\ (\beta \in B)\}$$

where $A = \{1, \ldots, p\}$, $B = \{p + 1, \ldots, m\}$.

We shall first show how the theory of the previous case can be applied to this problem. For all $x_0 \in S$ define the set of *active indices* at x_0 by

$$I(x_0) := \{\alpha \in A \mid g_\alpha(x_0) = 0\}$$

and let

$$S(x_0) := \{x \in \mathbf{R}^n \mid g_\alpha(x) = 0\ (\alpha \in I(x_0)),$$
$$g_\beta(x) = 0\ (\beta \in B)\}$$

together with its corresponding set of tangential constraints at x_0, that is,

$$R_{S(x_0)}(x_0) := \{h \in \mathbf{R}^n \mid g'_i(x_0; h) = 0$$
$$(i \in I(x_0) \cup B)\}.$$

Suppose x_0 affords a local minimum to f on S. If $g_\alpha(x_0) < 0$, let $\epsilon_\alpha > 0$ be such that $|x - x_0| < \epsilon_\alpha \Rightarrow g_\alpha(x) < 0$ and let

$$N(x_0) := \{x \in \mathbf{R}^n : |x - x_0| < \epsilon\}$$

where $\epsilon = \min\{\epsilon_\alpha \mid g_\alpha(x_0) < 0\}$. If $A = I(x_0)$, set $N(x_0) := \mathbf{R}^n$.

Since $S(x_0) \cap N(x_0) \subset S$, x_0 also affords a local minimum to f on $S(x_0)$. By Theorem 1.2, if x_0 is a normal point of $S(x_0)$ (that is, the linear equations $g'_i(x_0; h) = 0$ $(i \in I(x_0) \cup B)$ in h are linearly independent), then there exists a unique $\lambda \in \mathbf{R}^q$ (q denotes the cardinality of $I(x_0) \cup B$) such that $G_x(x_0, \lambda) = 0$ where, for all $((x, \lambda) \in \mathbf{R}^n \times \mathbf{R}^q)$,

$$G(x, \lambda) := f(x) + \sum_{i \in I(x_0) \cup B} \lambda_i g_i(x).$$

Moreover, for all $h \in R_{S(x_0)}(x_0)$,

$$\langle h, G_{xx}(x_0, \lambda)h \rangle \geq 0.$$

It can be shown that, in this event, $\lambda_\alpha \geq 0$ for all $\alpha \in I(x_0)$ and so, if we let

$$P(x_0) := \{\lambda \in \mathbf{R}^m \mid \lambda_\alpha \geq 0\ (\alpha \in I(x_0)),$$
$$\lambda_\alpha = 0\ (g_\alpha(x_0) < 0)\}$$

and define F as before, we obtain the following set of first and second order necessary conditions.

Theorem 1.3 *Suppose x_0 affords a local minimum to f on S. If x_0 is a normal point of $S(x_0)$ then there exists a unique $\lambda \in P(x_0)$ such that $F_x(x_0, \lambda) = 0$. Moreover, $\langle h, F_{xx}(x_0, \lambda)h \rangle \geq 0$ for all $h \in R_{S(x_0)}(x_0)$.*

This result does provide second order necessary conditions but, as we shall see next, they can be improved considerably. To do so, let us consider not curvilinear tangent vectors but the *tangent cone of S at x_0*, denoted by $T_S(x_0)$, which is the (closed) cone determined by the unit vectors h for which there exists a sequence $\{x_m\}$ in S converging to x_0 in the direction h in the sense that $x_m \neq x_0$, and

$$\lim_{m \to \infty} |x_m - x_0| = 0, \qquad \lim_{m \to \infty} \frac{x_m - x_0}{|x_m - x_0|} = h.$$

Define the set of vectors satisfying the *tangential constraints of S at x_0* by

$$R_S(x_0) := \{h \in \mathbf{R}^n \mid g'_\alpha(x_0; h) \leq 0\ (\alpha \in I(x_0)),$$
$$g'_\beta(x_0; h) = 0\ (\beta \in B)\}.$$

As before we have $T_S(x_0) \subset R_S(x_0)$ for all $x_0 \in S$, but the converse may not hold. If $T_S(x_0) = R_S(x_0)$, we say that $x_0 \in S$ is a *regular point of* S. Let

$$\mathcal{E} := \{x_0 \in S \mid \text{there exists } \lambda \in P(x_0) \text{ such that}$$
$$F_x(x_0, \lambda) = 0\}.$$

It is well-known that $x_0 \in \mathcal{E}$ if and only if

$$f'(x_0; h) \geq 0 \text{ for all } h \in R_S(x_0)$$

and, if x_0 is a regular point of S which affords a local minimum to f on S, then $x_0 \in \mathcal{E}$.

Now, suppose $x_0 \in \mathcal{E}$ and $\lambda \in P(x_0)$ is such that $F_x(x_0, \lambda) = 0$. Let $\Gamma := \{\alpha \in A \mid \lambda_\alpha > 0\}$ and define the set $\tilde{S}_\lambda$ of *modified constraints* as

$$\tilde{S}_\lambda := \{x \in \mathbf{R}^n \mid g_\alpha(x) \leq 0 \ (\alpha \in A,\ \lambda_\alpha = 0),$$
$$g_\beta(x) = 0 \ (\beta \in \Gamma \cup B)\}$$

which satisfies

$$\begin{aligned} \tilde{S}_\lambda &= \{x \in S \mid g_\alpha(x) = 0 \ (\alpha \in \Gamma)\} \\ &= \{x \in S \mid F(x, \lambda) = f(x)\}. \end{aligned}$$

Define a set of *modified tangential constraints* as

$$\tilde{R}_S(x_0; \lambda) := \{h \in \mathbf{R}^n \mid g'_\alpha(x_0; h) \leq 0$$
$$(\alpha \in I(x_0),\ \lambda_\alpha = 0),\ g'_\beta(x_0; h) = 0 \ (\beta \in \Gamma \cup B)\}$$

which satisfies

$$\tilde{R}_S(x_0; \lambda) = \{h \in R_S(x_0) \mid g'_\alpha(x_0; h) = 0 \ (\alpha \in \Gamma)\}$$
$$= \{h \in R_S(x_0) \mid f'(x_0; h) = 0\}.$$

The improved second order conditions correspond to the following result.

Theorem 1.4 *Suppose x_0 affords a local minimum to f on S and $x_0 \in \mathcal{E}$. Let $\lambda \in P(x_0)$ be such that $F_x(x_0, \lambda) = 0$. If x_0 is a regular point of $\tilde{S}_\lambda$ then $\langle h, F_{xx}(x_0, \lambda)h \rangle \geq 0$ for all $h \in \tilde{R}_S(x_0; \lambda)$.*

Let us briefly mention some simple criteria for regularity. Given x_0 in S, the following are equivalent:

a. $\{g'_\beta(x_0) \mid \beta \in B\}$ is linearly independent and, if $p > 0$, there exists $h \in \mathbf{R}^n$ such that

$$g'_\alpha(x_0; h) < 0 \ (\alpha \in I(x_0)),\ g'_\beta(x_0; h) = 0 \ (\beta \in B).$$

b. The relations $\sum_1^m \lambda_i g'_i(x_0) = 0$ and $\lambda \in P(x_0)$ imply that $\lambda = 0$.

These conditions imply that x_0 is a regular point of S, and x_0 is said to be *normal* if it satisfies (a) or (b).

Note that, for any $\lambda \in P(x_0)$, $R_{S(x_0)}(x_0)$ is a subset of $\tilde{R}_S(x_0; \lambda)$, and usually the contention is proper. In fact, one can easily find examples for which a point x_0 belongs to $\mathcal{E}$ so that, for some $\lambda \in P(x_0)$, $F_x(x_0, \lambda) = 0$ and, moreover, $\langle h, F_{xx}(x_0, \lambda)h \rangle \geq 0$ for all $h \in R_{S(x_0)}(x_0)$, but x_0 is a regular point $\tilde{S}_\lambda$ and $\langle h, F_{xx}(x_0, \lambda)h \rangle < 0$ for some $h \in \tilde{R}_S(x_0; \lambda)$. In this event, Theorem 1.3 gives no information, but one concludes from Theorem 1.4 that the point x_0 does not afford a local minimum to f on S. For this reason, we shall call the second order conditions given in Theorems 1.3 and 1.4 *weak* and *strong* conditions respectively.

B. Isoperimetric control problem of Lagrange

Let us consider now the following optimal control problem involving isoperimetric constraints (see [6] for details), and briefly explain how a situation similar to that of the finite dimensional case occurs.

Suppose we are interested in minimizing

$$I(x, u) = \int_{t_0}^{t_1} L(t, x(t), u(t))dt$$

subject to

$x: T \to \mathbf{R}^n$ piecewise C^1; $u: T \to \mathbf{R}^m$ piecewise continuous;

$\dot{x}(t) = f(t, x(t), u(t))$ $(t \in T)$, and $x(t_0) = \xi_0$, $x(t_1) = \xi_1$;

$I_\alpha(x, u) \leq 0$ $(\alpha \in R)$, $I_\beta(x, u) = 0$ $(\beta \in Q)$,

where R and Q are two disjoint index sets, $T = [t_0, t_1]$, and

$$I_\gamma(x, u) := \alpha_\gamma + \int_{t_0}^{t_1} L_\gamma(t, x(t), u(t))dt.$$

Assuming that f, L, L_γ are C^2, the strong second order conditions established in [6] state that, if (x_0, u_0) is an “extremal” which solves the problem (with, for simplicity, all indices active) then, under certain “normality” assumptions, a quadratic form is nonnegative on the set of all (y, v) satisfying

i. $\dot{y}(t) = A(t)y(t) + B(t)v(t)$ $(t \in T)$, and $y(t_0) = y(t_1) = 0$;

ii. $I'_\alpha((x_0, u_0); (y, v)) \leq 0$ ($\alpha \in R$ with $\mu_\alpha = 0$);

iii. $I'_\beta((x_0, u_0); (y, v)) = 0$ ($\beta \in R$ with $\mu_\beta > 0$, or $\beta \in Q$)

where A and B are given by $A(t) = f_x(t, x_0(t), u_0(t))$ and $B(t) = f_u(t, x_0(t), u_0(t))$.

The weak version of this result states that the above relation holds for all (y, v) satisfying (i) and

$$I'_\gamma((x_0, u_0); (y, v)) = 0 \quad (\gamma \in R \cup Q)$$

(instead of (ii) and (iii)).

C. Mayer problem with endpoint constraints

A similar situation occurs with the optimal control problem considered in [5] where one is interested in minimizing a functional $\phi_0(x(t_0), x(t_1))$ subject to $\dot{x}(t) = f(t, x(t), u(t))$ a.e. in T and constraints of the form

$$\phi_\alpha(x(t_0), x(t_1)) \leq 0 \; (\alpha \in R),$$
$$\phi_\beta(x(t_0), x(t_1)) = 0 \; (\beta \in Q).$$

First and second order conditions are derived, without normality assumptions, from results obtained for an abstract optimization problem. When certain normality assumptions are imposed, however, second order conditions are expressed in terms of solutions (η, ω) to the linear system $\dot{\eta}(t) = \bar{f}_x(t)\eta(t) + \bar{f}_u(t)\omega(t)$ satisfying strong conditions, as in the previous two cases, given by

$$\bar{\phi}_{ix}\eta(t_0) + \bar{\phi}_{iy}\eta(t_1) \leq 0 \quad (i \in I_A,\; \lambda_i = 0),$$

$$\bar{\phi}_{ix}\eta(t_0) + \bar{\phi}_{iy}\eta(t_1) = 0 \; (i \in I_A,\; \lambda_i > 0, \text{ or } i \in Q),$$

where I_A denotes the set of active inequality constraints. Again, the weak version would produce solutions to the linear system satisfying

$$\bar{\phi}_{ix}\eta(t_0) + \bar{\phi}_{iy}\eta(t_1) = 0 \quad (i \in I_A \cup Q).$$

D. Lagrange problem with control constraints

In this paper we shall be concerned with a Lagrange control problem posed over piecewise continuous controls and such that the control function u is restricted to satisfy

$$\varphi_\alpha(u(t)) \leq 0 \; (\alpha \in R), \quad \varphi_\beta(u(t)) = 0 \; (\beta \in Q).$$

The techniques used in [6] for the isoperimetric problem of Lagrange, or in [5] for the Mayer problem with endpoint constraints, do not apply to this problem since they are essentially pointwise in nature, while the problem we shall deal with involves constraints which affect the whole underlying time interval.

In the literature, one can find different derivations of second order conditions for such a problem (see, for example, [4, 8, 13, 15–17]) and the conditions one encounters in those references are of the weak type. Let us point out that, in [8, 15], the main results on second order conditions are not proved and, quoting [8], "the derivation of the conditions is very special and difficult."

An exception is to be found in [9, 10] where a set of "modified admissible variations" for the problem in hand was proposed, thus yielding strong second order conditions. However, the technique used in [9, 10] requires certain crucial assumptions on the data of the problem such as convexity of the control set and, even in that case, the conditions are shown to hold only for particular problems.

Let us finally mention that an entirely different approach for optimal control problems involving equality constraints can be found, for example, in [20], where results from abstract optimization theory on Banach spaces are applied to the optimal control problem posed over L^∞-controls, a technique which does not work in our setting. In more recent works (see [1, 2]) a special emphasis has been laid on conditions without a priori normality assumptions, and powerful new techniques such as that of using the normal cone introduced by Mordukhovich [14] have produced important contributions to the subject, but no strong conditions of the type we refer to are present.

In Section 2 we state the optimal control problem we shall be concerned with as well as first and second order conditions and the notions of "extremal" and "strong normality." In Section 3 we introduce a weak notion of normality together with three different notions of regularity in terms of certain convex cones, and show the relation between normality and regularity. Section 4 includes four examples for which the corresponding quadratic form may be negative on different cones of critical directions.

II. Statement of the Problem

In this section we shall pose the problem we shall be dealing with together with some results on first and second order necessary conditions which form the frame of the questions related to the sign of a quadratic form.

Suppose we are given an interval $T := [t_0, t_1]$ in $\mathbf{R}$, two points ξ_0, ξ_1 in $\mathbf{R}^n$, and functions L and f mapping $T \times \mathbf{R}^n \times \mathbf{R}^m$ to $\mathbf{R}$ and $\mathbf{R}^n$ respectively, and $\varphi = (\varphi_1, \ldots, \varphi_q)$ mapping $\mathbf{R}^m$ to $\mathbf{R}^q$ $(q \leq m)$. Let

$$U := \{u \in \mathbf{R}^m \mid \varphi_\alpha(u) \leq 0 \; (\alpha \in R),$$
$$\varphi_\beta(u) = 0 \; (\beta \in Q)\}$$

where $R = \{1, \ldots, r\}$, $Q = \{r+1, \ldots, q\}$. Denote by X the space of piecewise C^1 functions mapping T to $\mathbf{R}^n$, by $\mathcal{U}$ the space of piecewise continuous functions mapping T to $\mathbf{R}^m$, set $Z := X \times \mathcal{U}$,

$$D := \{(x, u) \in Z \mid \dot{x}(t) = f(t, x(t), u(t)) \; (t \in T)\},$$

$$Z_e(U) := \{(x, u) \in D \mid u(t) \in U \; (t \in T),$$
$$x(t_0) = \xi_0,\; x(t_1) = \xi_1\},$$

and consider the functional $I\colon Z \to \mathbf{R}$ given by

$$I(x, u) := \int_{t_0}^{t_1} L(t, x(t), u(t))dt \quad ((x, u) \in Z).$$

The problem we shall deal with, which we label (P), is that of minimizing I over $Z_e(U)$.

A common and concise way of formulating this problem is as follows:

Minimize $I(x,u) = \int_{t_0}^{t_1} L(t,x(t),u(t))dt$ subject to

$x{:}T \to \mathbf{R}^n$ piecewise C^1; $u{:}T \to \mathbf{R}^m$ piecewise continuous;

$\dot{x}(t) = f(t,x(t),u(t))$ $(t \in T)$, and $x(t_0) = \xi_0$, $x(t_1) = \xi_1$;

$\varphi_\alpha(u(t)) \le 0$ and $\varphi_\beta(u(t)) = 0$ $(\alpha \in R,\ \beta \in Q,\ t \in T)$.

We have chosen this fixed-endpoint control problem of Lagrange for simplicity of exposition, and to keep notational complexity to a minimum, but no difficulties arise in extending the theory to follow to Bolza problems with possible variable endpoints.

Elements of Z will be called *processes*, of $Z_e(U)$ *admissible processes*, and a process (x,u) *solves* (P) if (x,u) is admissible and $I(x,u) \le I(y,v)$ for all admissible process (y,v). For any $(x,u) \in Z$ we use the notation $(\tilde{x}(t))$ to represent $(t,x(t),u(t))$, and '$*$' denotes transpose. We assume that L, f and φ are C^2 and the $q \times (m+r)$-dimensional matrix

$$\left(\frac{\partial \varphi_i}{\partial u^k} \ \ \delta_{i\alpha}\varphi_\alpha \right)$$

$(i = 1,\ldots,q;\ \alpha = 1,\ldots,r;\ k = 1,\ldots,m)$ has rank q on U (here $\delta_{\alpha\alpha} = 1$, $\delta_{\alpha\beta} = 0$ $(\alpha \ne \beta)$). This condition is equivalent to the condition that, at each point u in U, the matrix

$$\left(\frac{\partial \varphi_i}{\partial u^k} \right) \quad (i = i_1,\ldots,i_p;\ k = 1,\ldots,m)$$

has rank p, where $i_1,\ldots,i_p$ are the indices $i \in R \cup Q$ such that $\varphi_i(u) = 0$ (see [1] for details).

First order conditions for this problem are well established (see, for example, [3, 6, 9]), and one version can be written as follows. For all (t,x,u,p,μ,λ) in $T \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^q \times \mathbf{R}$ let

$$H(t,x,u,p,\mu,\lambda) := \langle p, f(t,x,u)\rangle$$
$$-\lambda L(t,x,u) - \langle \mu, \varphi(u)\rangle,$$

and denote by $\mathcal{U}_q$ the space of all piecewise continuous functions mapping T to $\mathbf{R}^q$.

Theorem 2.1 *Suppose* (x_0,u_0) *solves* (P). *Then there exist* $\lambda_0 \ge 0$, $p \in X$, *and* $\mu \in \mathcal{U}_q$ *continuous on each interval of continuity of* u_0, *not vanishing simultaneously on* T, *such that*

a. $\mu_\alpha(t) \ge 0$ $(\alpha \in R,\ t \in T)$ *with* $\mu_\alpha(t) = 0$ *whenever* $\varphi_\alpha(u_0(t)) < 0$.

b. *On every interval of continuity of* u_0,

$$\dot{p}(t) = -H_x^*(\tilde{x}_0(t),p(t),\mu(t),\lambda_0),$$
$$H_u(\tilde{x}_0(t),p(t),\mu(t),\lambda_0) = 0.$$

c. *For all* $(t,u) \in T \times U$,

$$H(t,x_0(t),u,p(t),0,\lambda_0) \le H(\tilde{x}_0(t),p(t),0,\lambda_0).$$

Note that (a) and (c) are equivalent, respectively, to the following conditions:

a. $\mu_\alpha(t) \ge 0$ and $\mu_\alpha(t)\varphi_\alpha(u_0(t)) = 0$ $(\alpha \in R,\ t \in T)$;

c. $H(t,x_0(t),u,p(t),\mu(t),\lambda_0) + \langle \mu(t),\varphi(u)\rangle \le H(\tilde{x}_0(t),p(t),\mu(t),\lambda_0)$ for all $(t,u) \in T \times U$.

Based on this theorem, let us introduce a set $M(x,u)$ of multipliers together with a set $\mathcal{E}$ whose elements, which will be called "extremals," have associated a nonzero cost multiplier normalized to one.

Definition 2.2 For all $(x,u) \in Z$ let $M(x,u)$ be the set of all $(p,\mu,\lambda_0) \in X \times \mathcal{U}_q \times \mathbf{R}$ with $\lambda_0 + |p| \ne 0$ satisfying

a. $\mu_\alpha(t) \ge 0$, $\mu_\alpha(t)\varphi_\alpha(u(t)) = 0$ for all $\alpha \in R$, $t \in T$.

b. $\dot{p}(t) = -H_x^*(\tilde{x}(t),p(t),\mu(t),\lambda_0)$ $(t \in T)$.

c. $H_u(\tilde{x}(t),p(t),\mu(t),\lambda_0) = 0$ $(t \in T)$.

Denote by $\mathcal{E}$ be the set of all $(x,u,p,\mu) \in Z \times X \times \mathcal{U}_q$ such that $(p,\mu,1) \in M(x,u)$, that is,

a. $\mu_\alpha(t) \ge 0$, $\mu_\alpha(t)\varphi_\alpha(u(t)) = 0$ for all $\alpha \in R$, $t \in T$.

b. $\dot{p}(t) = -f_x^*(\tilde{x}(t))p(t) + L_x^*(\tilde{x}(t))$ $(t \in T)$.

c. $f_u^*(\tilde{x}(t))p(t) = L_u^*(\tilde{x}(t)) + \varphi'^*(u(t))\mu(t)$ for all $t \in T$.

The notion of "strong normality," as defined below, is introduced to assure that, if (p,μ,λ_0) is a triple of multipliers corresponding to a strongly normal solution to the problem, then $\lambda_0 > 0$ and, when $\lambda_0 = 1$, the pair (p,μ) is unique.

Definition 2.3 A process (x,u) will be said to be *strongly normal* if, given $p \in X$ and $\mu \in \mathcal{U}_q$ satisfying

i. $\mu_\alpha(t)\varphi_\alpha(u(t)) = 0$ $(\alpha \in R,\ t \in T)$;

ii. $\dot{p}(t) = -f_x^*(\tilde{x}(t))p(t)$
$[\, = -H_x^*(\tilde{x}(t),p(t),\mu(t),0)\,]$ $(t \in T)$;

iii. $0 = f_u^*(\tilde{x}(t))p(t) - \varphi'^*(u(t))\mu(t)$
$[\, = H_u^*(\tilde{x}(t),p(t),\mu(t),0)\,]$ $(t \in T)$,

then $p \equiv 0$. In this event, clearly, also $\mu \equiv 0$.

Proposition 2.4 *If* (x,u) *solves* (P) *then* $M(x,u) \ne \emptyset$. *If also* (x,u) *is strongly normal then there exists a unique* $(p,\mu) \in X \times \mathcal{U}_q$ *such that* $(x,u,p,\mu) \in \mathcal{E}$.

Proof: Let (x, u) solve (P). By Theorem 2.1 there exists $(p, \mu, \lambda_0) \in M(x, u)$. Suppose (x, u) is strongly normal. Clearly we have $\lambda_0 \neq 0$ and, if $(q, \nu, \lambda_0) \in M(x, u)$, then

i. $[\mu_\alpha(t) - \nu_\alpha(t)]\varphi_\alpha(u(t)) = 0$ $(\alpha \in R,\ t \in T)$;
ii. $[\dot{p}(t) - \dot{q}(t)] = -f_x^*(\tilde{x}(t))[p(t) - q(t)]$ $(t \in T)$;
iii. $f_u^*(\tilde{x}(t))[p(t) - q(t)] - \varphi'^*(u(t))[\mu(t) - \nu(t)] = 0$ $(t \in T)$,

implying that $p \equiv q$ and $\mu \equiv \nu$. The result follows by choosing $\lambda_0 = 1$ since $(p/\lambda_0, \mu/\lambda_0, 1) \in M(x, u)$. ∎

For any $(x, u, p, \mu) \in Z \times X \times \mathcal{U}_q$ and $(y, v) \in Z$, let us consider the following quadratic form:

$$J((x, u, p, \mu); (y, v)) := \int_{t_0}^{t_1} 2\Omega(t, y(t), v(t))dt$$

where, for all $(t, y, v) \in T \times \mathbf{R}^n \times \mathbf{R}^m$,

$$2\Omega(t, y, v) := -[\langle y, H_{xx}(t)y\rangle + 2\langle y, H_{xu}(t)v\rangle + \langle v, H_{uu}(t)v\rangle]$$

and $H(t)$ denotes $H(\tilde{x}(t), p(t), \mu(t), 1)$.

For all $u \in \mathbf{R}^m$ define the set of *active indices at* u as

$$I_a(u) := \{\alpha \in R \mid \varphi_\alpha(u) = 0\}.$$

As mentioned in the introduction, a set of weak second order conditions for problem (P) can be found in the literature. In particular, the following result was derived in [4] by reducing the original problem into a problem involving only equality constraints in the control.

Theorem 2.5 *Let (x_0, u_0) be an admissible process for which there exists $(p, \mu) \in X \times \mathcal{U}_q$ such that $(x_0, u_0, p, \mu) \in \mathcal{E}$. If (x_0, u_0) is a strongly normal solution to* (P) *then*

$$J((x_0, u_0, p, \mu); (y, v)) \geq 0$$

for all $(y, v) \in Z$ satisfying

i. $\dot{y}(t) = f_x(\tilde{x}_0(t))y(t) + f_u(\tilde{x}_0(t))v(t)$ $(t \in T)$, *and* $y(t_0) = y(t_1) = 0$;
ii. $\varphi_i'(u_0(t))v(t) = 0$ $(i \in I_a(u_0(t)) \cup Q,\ t \in T)$.

The same cone of critical directions or "admissible variations" defined by (ii) yields second order necessary conditions in other references mentioned in the introduction. Those conditions are obtained in different ways and, in some cases, under different assumptions, but they are all expressed in terms of that set of variations. Let us briefly mention that the same device used in [3], which consists in defining the functions

$$\psi_\alpha(u, w) := \varphi_\alpha(u) + (w^\alpha)^2 \ \ (\alpha \in R),$$

$$\psi_\beta(u, w) = \varphi_\beta(u) \ \ (\beta \in Q),$$

appears in [17] together with an application of the results obtained in [16].

III. Normality and Regularity

It is of interest to see if the assumption of strong normality, which implies uniqueness of the pair (p, μ) such that (x_0, u_0, p, μ) is an extremal as well as the set of second order conditions given in Theorem 2.5, can be weakened.

To do so, we shall first compare that notion with a different one used in other references (see [9, 10]), and characterize those notions in terms of certain convex cones.

Recall that, if $A(t) := f_x(\tilde{x}(t))$ and $B(t) := f_u(\tilde{x}(t))$, (x, u) is *strongly normal* if, given $(p, \mu) \in X \times \mathcal{U}_q$ satisfying

i. $\mu_\alpha(t)\varphi_\alpha(u(t)) = 0$ $(\alpha \in R,\ t \in T)$;
ii. $\dot{p}(t) = -A^*(t)p(t)$ $(t \in T)$;
iii. $B^*(t)p(t) = \varphi'^*(u(t))\mu(t)$ $(t \in T)$,

then $p \equiv 0$.

This notion can be characterized in terms of a subspace of $\mathbf{R}^m$ as follows.

Definition 3.1 For any $u \in \mathbf{R}^m$ let

$$\tau_0(u) := \{h \in \mathbf{R}^m \mid \varphi_i'(u)h = 0 \ (i \in I_a(u) \cup Q)\}.$$

A process (x, u) will be said to be τ_0-*regular* if there is no nonnull solution $z \in X$ to the system

$$\dot{z}(t) = -A^*(t)z(t),$$

$$z^*(t)B(t)h = 0 \text{ for all } h \in \tau_0(u(t)) \ (t \in T).$$

Proposition 3.2 *For any $(x, u) \in Z(U)$ the following are equivalent:*

a. *(x, u) is τ_0-regular.*
b. *(x, u) is strongly normal.*

Proof: (a) ⇒ (b): Suppose $(p, \mu) \in X \times \mathcal{U}_q$ is such that $\mu_\alpha(t)\varphi_\alpha(u(t)) = 0$ $(\alpha \in R,\ t \in T)$ and

$$\dot{p}(t) = -A^*(t)p(t),\ B^*(t)p(t) = \varphi'^*(u(t))\mu(t).$$

Let $h \in \tau_0(u(t))$. Then

$$p^*(t)B(t)h = \sum_{i=1}^{q} \mu_i(t)\varphi_i'(u(t))h = 0$$

and so, by (a), $p \equiv 0$.

(b) ⇒ (a): Let $z \in X$ be such that

$$\dot{z}(t) = -A^*(t)z(t),$$

$$z^*(t)B(t)h = 0 \text{ for all } h \in \tau_0(u(t)) \ (t \in T).$$

For each $t \in T$ let $\hat{\varphi} = (\varphi_{i_1}, \dots, \varphi_{i_p})$ where

$$I_a(u(t)) \cup Q = \{i_1, \dots, i_p\}$$

and define $\hat{\mu}(t) = (\hat{\mu}_{i_1}(t), \dots, \hat{\mu}_{i_p}(t))$ by

$$\hat{\mu}(t) := \Lambda^{-1}(t)\hat{\varphi}'(u(t))B^*(t)z(t) \quad (t \in T)$$

where $\Lambda(t) = \hat{\varphi}'(u(t))\hat{\varphi}'^*(u(t))$. Note that, since

$$\hat{\varphi}'(u(t))\hat{\varphi}'^*(u(t))\Lambda^{-1}(t) =$$

$$\Lambda^{-1}(t)^*\hat{\varphi}'(u(t))\hat{\varphi}'^*(u(t)) = I_{p\times p}$$

we have $\Lambda^{-1}(t) = \Lambda^{-1}(t)^*$ $(t \in T)$. Let $\mu(t) = (\mu_1(t), \dots, \mu_q(t))$ where

$$\mu_\alpha(t) := \begin{cases} \hat{\mu}_{i_r}(t) & \text{if } \alpha = i_r,\ r = 1, \dots, p \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\mu_\alpha(t)\varphi_\alpha(u(t)) = 0$ $(\alpha \in R,\ t \in T)$ and

$$\hat{\mu}^*(t)\hat{\varphi}'(u(t)) = \mu^*(t)\varphi'(u(t)) \quad (t \in T).$$

Now, let

$$G(t) := I_{m\times m} - \hat{\varphi}'^*(u(t))\Lambda^{-1}(t)\hat{\varphi}'(u(t))$$

and note that $\hat{\varphi}'(u(t))G(t) = 0$ $(t \in T)$. If $h_k(t)$ $(k = 1, \dots, m)$ denotes the k-th column of $G(t)$, we have

$$\varphi'_{i_j}(u(t))h_k(t) = 0 \quad (j = 1, \dots, p,\ k = 1, \dots, m)$$

that is, $h_k(t)$ belongs to $\tau_0(u(t))$ and, therefore,

$$z^*(t)B(t)h_k(t) = 0 \quad (k = 1, \dots, m).$$

Thus

$$0 = z^*(t)B(t)G(t) = z^*(t)B(t) - \mu^*(t)\varphi'(u(t))$$

and so, by (b), $z \equiv 0$. ∎

Let us now turn to a different notion of normality. Note that the sign of $\mu_\alpha(t)$ is not considered in the definition of strong normality. By adding such condition, we obtain a weaker notion of normality.

Definition 3.3 We shall say that $(x, u) \in Z$ is *weakly normal* if, given $(p, \mu) \in X \times \mathcal{U}_q$ satisfying

i. $\mu_\alpha(t) \geq 0$, $\mu_\alpha(t)\varphi_\alpha(u(t)) = 0$ $(\alpha \in R,\ t \in T)$;
ii. $\dot{p}(t) = -A^*(t)p(t)$ $(t \in T)$;
iii. $B^*(t)p(t) = \varphi'^*(u(t))\mu(t)$ $(t \in T)$,

then $p \equiv 0$.

Observe that in Proposition 2.4, if we replace the assumption of strong with that of weak normality, the result remains valid except for the uniqueness of (p, μ).

For all $\mu \in \mathbf{R}^q$ define the following subsets of indices of R:

$$\Gamma_0(\mu) := \{\alpha \in R \mid \mu_\alpha = 0\},$$

$$\Gamma_p(\mu) := \{\alpha \in R \mid \mu_\alpha > 0\},$$

and let us consider the following convex cones of $\mathbf{R}^m$.

Definition 3.4 For any $u \in \mathbf{R}^m$ and $\mu \in \mathbf{R}^q$ let

$$\tau_1(u, \mu) := \{h : \varphi'_i(u)h \leq 0 \ (i \in I_a(u) \cap \Gamma_0(\mu)),$$
$$\varphi'_j(u)h = 0 \ (j \in \Gamma_p(\mu) \cup Q)\}$$
$$\tau_2(u) := \{h : \varphi'_i(u)h \leq 0 \ (i \in I_a(u)),$$
$$\varphi'_j(u)h = 0 \ (j \in Q)\}.$$

Definition 3.5 Let $(x, u) \in Z(U)$ and $\mu \in \mathcal{U}_q$ with $\mu_\alpha(t) \geq 0$ and $\mu_\alpha(t)\varphi_\alpha(u(t)) = 0$ $(\alpha \in R,\ t \in T)$.

a. We say (x, u, μ) is τ_1*-regular* if there is no nonnull solution $z \in X$ to the system

$$\dot{z}(t) = -A^*(t)z(t),$$

$$z^*(t)B(t)h \leq 0 \text{ for all } h \in \tau_1(u(t), \mu(t)) \ (t \in T).$$

b. We say (x, u) is τ_2*-regular* if there is no nonnull solution $z \in X$ to the system

$$\dot{z}(t) = -A^*(t)z(t),$$

$$z^*(t)B(t)h \leq 0 \text{ for all } h \in \tau_2(u(t)) \ (t \in T).$$

Now, as one readily verifies, τ_0-regularity implies τ_1-regularity which in turn implies τ_2-regularity. Let us give a formal proof of this fact.

Proposition 3.6 *Let* $(x, u) \in Z$ *and* $\mu \in \mathcal{U}_q$ *with* $\mu_\alpha(t) \geq 0$ *and* $\mu_\alpha(t)\varphi_\alpha(u(t)) = 0$ $(\alpha \in R,\ t \in T)$, *and consider the following statements:*

a. (x, u) *is* τ_0*-regular.*
b. (x, u, μ) *is* τ_1*-regular.*
c. (x, u) *is* τ_2*-regular.*

Then (a) ⇒ (b) ⇒ (c).

Proof: (a) ⇒ (b): Let $z \in X$ be such that

$$\dot{z}(t) = -A^*(t)z(t),$$

$$z^*(t)B(t)h \leq 0 \text{ for all } h \in \tau_1(u(t), \mu(t)) \ (t \in T).$$

Let $h \in \tau_0(u(t))$. If $i \in I_a(u(t)) \cap \Gamma_0(\mu(t))$ then we have $\varphi'_i(u(t))h = 0$ and, if $j \in \Gamma_p(\mu(t)) \cup Q$ then

$\varphi_j'(u(t))h = 0$ since $\Gamma_p(\mu(t)) \subset I_a(u(t))$. This shows that $h \in \tau_1(u(t), \mu(t))$ and so

$$\tau_0(u(t)) \subset \tau_1(u(t), \mu(t)).$$

By assumption, $z^*(t)B(t)h \leq 0$. However, since $\tau_0(u(t))$ is a subspace, we have $z^*(t)B(t)h = 0$. By (a) this implies that $z \equiv 0$ and this proves (b).

(b) $\Rightarrow$ (c): Let $z \in X$ be such that

$$\dot{z}(t) = -A^*(t)z(t),$$

$$z^*(t)B(t)h \leq 0 \text{ for all } h \in \tau_2(u(t)) \ (t \in T).$$

Let $h \in \tau_1(u(t), \mu(t))$. If $i \in I_a(u(t))$ then

$$\varphi_i'(u(t))h \leq 0 \text{ if } \mu_i(t) = 0,$$

$$\varphi_i'(u(t))h = 0 \text{ if } \mu_i(t) > 0.$$

Also $\varphi_j'(u(t))h = 0$ if $j \in Q$. Therefore $h \in \tau_2(u(t))$ and so $\tau_1(u(t), \mu(t)) \subset \tau_2(u(t))$. By (b), $z \equiv 0$ and (c) follows. ∎

Let us now prove that, just as τ_0-regularity is equivalent to strong normality, the same occurs with respect to τ_2-regularity and weak normality.

Proposition 3.7 *For any $(x, u) \in Z(U)$ the following are equivalent:*

a. *(x, u) is τ_2-regular.*

b. *(x, u) is weakly normal.*

Proof: (a) $\Rightarrow$ (b): Suppose $(p, \mu) \in X \times \mathcal{U}_q$ is such that $\mu_\alpha(t) \geq 0$, $\mu_\alpha(t)\varphi_\alpha(u(t)) = 0$ $(\alpha \in R,\ t \in T)$,

$$\dot{p}(t) = -A^*(t)p(t),\ B^*(t)p(t) = \varphi'^*(u(t))\mu(t).$$

Let $h \in \tau_2(u(t))$. Using the fact that $\mu_\alpha(t) \geq 0$ $(\alpha \in R)$ and $\mu_\alpha(t) = 0$ whenever $\varphi_\alpha(u(t)) < 0$, we have

$$p^*(t)B(t)h = \sum_{i=1}^{q} \mu_i(t)\varphi_i'(u(t))h \leq 0$$

implying, by (a), that $p \equiv 0$.

(b) $\Rightarrow$ (a): Let $z \in X$ be such that

$$\dot{z}(t) = -A^*(t)z(t),$$

$$z^*(t)B(t)h \leq 0 \text{ for all } h \in \tau_2(u(t)) \ (t \in T).$$

Proceed as in the proof of (b) $\Rightarrow$ (a) of Proposition 3.2 showing that $h_k(t)$, as defined in that proof, belongs to $\tau_0(u(t))$. Now, since $\tau_0(u(t)) \subset \tau_2(u(t))$, our assumptions on z imply that $z^*(t)B(t)h_k(t) \leq 0$ $(k = 1, \ldots, m)$. However, also $-h_k(t) \in \tau_0(u(t))$ and so $z^*(t)B(t)h_k(t) = 0$ $(k = 1, \ldots, m)$. Thus

$$0 = z^*(t)B(t)G(t) = z^*(t)B(t) - \mu^*(t)\varphi'(u(t)).$$

The result will follow by (b) if $\mu_\alpha(t) \geq 0$ $(\alpha \in R,\ t \in T)$. To prove that this is indeed the case, let $C(t)$ be the $p \times m$ matrix $C(t) := \Lambda^{-1}(t)\hat{\varphi}'(u(t))$ and observe that

$$C(t)\hat{\varphi}'^*(u(t)) = I_{p\times p} = \hat{\varphi}'(u(t))C^*(t).$$

Therefore, if $c_j(t)$ denotes the j-th column of $C^*(t)$ and $\{e_j\}$ the canonical base in $\mathbf{R}^p$ $(j = 1, \ldots, p)$ then, for $j = 1, \ldots, p$,

$$(\varphi_{i_j}'(u(t))c_1(t), \ldots, \varphi_{i_j}'(u(t))c_p(t)) = e_j^*.$$

Thus, if $j \in \{1, \ldots, p\}$ is such that $i_j \in I_a(u(t))$, then

$$\varphi_k'(u(t))(-c_j(t)) = \begin{cases} -1 & \text{if } k = i_j \\ 0 & \text{if } k \neq i_j \end{cases}$$

implying that $-c_j(t) \in \tau_2(u(t))$. Therefore

$$z^*(t)B(t)c_j(t) \geq 0 \quad (t \in T).$$

But $\hat{\mu}^*(t) = z^*(t)B(t)C^*(t)$ and so $\hat{\mu}_\alpha^*(t) \geq 0$ for all $\alpha \in I_a(u(t))$. ∎

IV. Examples

Theorem 2.5 can be stated in terms of $\tau_0(u_0(t))$. Explicitly, if (x_0, u_0) is a strongly normal solution to (P) and $(p, \mu) \in X \times \mathcal{U}_q$ is the unique pair such that $(x_0, u_0, p, \mu) \in \mathcal{E}$ then Theorem 2.5 states that $J((x_0, u_0, p, \mu); (y, v)) \geq 0$ for all $(y, v) \in Z$ satisfying

i. $\dot{y}(t) = f_x(\tilde{x}_0(t))y(t) + f_u(\tilde{x}_0(t))v(t)$ $(t \in T)$, and $y(t_0) = y(t_1) = 0$;

ii. $v(t) \in \tau_0(u_0(t))$ $(t \in T)$.

We begin this section by providing a weakly normal solution (x_0, u_0) to (P) with (p, μ) a pair such that $(x_0, u_0, p, \mu) \in \mathcal{E}$, but $J((x_0, u_0, p, \mu); (y, v)) < 0$ for some $(y, v) \in Z$ satisfying (i) and $v(t) \in \tau_2(u_0(t))$ $(t \in T)$.

Example 4.1 Consider the problem (P) of minimizing

$$I(x, u) = \int_0^1 u_1(t)dt$$

subject to

$$\dot{x}(t) = u_1^2(t) + u_2(t),$$

$$u_1(t) \geq 0,\ u_1(t) \geq u_2(t) \ (t \in [0, 1])$$

and $x(0) = x(1) = 0$.

In this case we have $T = [0, 1]$, $n = 1$, $m = r = q = 2$, $\xi_0 = \xi_1 = 0$ and, for any $t \in T$, $x \in \mathbf{R}$, and $u \in \mathbf{R}^2$ with $u = (u_1, u_2)$,

$$L(t, x, u) = u_1, \quad f(t, x, u) = u_1^2 + u_2,$$

$$\varphi_1(u) = -u_1, \quad \varphi_2(u) = u_2 - u_1.$$

Observe first that

$$H(t, x, u, p, \mu, 1) =$$

$$p(u_1^2 + u_2) - u_1 + (\mu_1 + \mu_2)u_1 - \mu_2 u_2$$

so that

$$H_u(t, x, u, p, \mu, 1) = (2pu_1 - 1 + \mu_1 + \mu_2, p - \mu_2),$$

$$H_{uu}(t, x, u, p, \mu, 1) = \begin{pmatrix} 2p & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, for any $(x, u, p, \mu) \in Z \times X \times \mathcal{U}_2$ and $(y, v) \in Z$,

$$J((x, u, p, \mu); (y, v)) = -\int_0^1 2p(t)v_1^2(t)dt.$$

Clearly $(x_0, u_0) \equiv (0, 0)$ solves the problem. Since $\varphi_1'(u_0(t)) = (-1, 0)$ and $\varphi_2'(u_0(t)) = (-1, 1)$, we have

$$\tau_0(u_0(t)) = \{h \in \mathbf{R}^2 \mid -h_1 = 0,\ -h_1 + h_2 = 0\}$$

$$\tau_1(u_0(t), \mu(t)) = \{h \in \mathbf{R}^2 \mid -h_1 \leq 0 \text{ if } \mu_1(t) = 0,$$
$$-h_1 = 0 \text{ if } \mu_1(t) > 0,$$
$$-h_1 + h_2 \leq 0 \text{ if } \mu_2(t) = 0,$$
$$-h_1 + h_2 = 0 \text{ if } \mu_2(t) > 0\},$$

$$\tau_2(u_0(t)) = \{h \in \mathbf{R}^2 \mid -h_1 \leq 0,\ -h_1 + h_2 \leq 0\}.$$

To test for regularity note that, since $f_x(\tilde{x}_0(t)) = 0$ and $f_u(\tilde{x}_0(t)) = (0, 1)$, the system

$$\dot{z}(t) = -A^*(t)z(t) = 0,$$

$$z^*(t)B(t)h = z(t)h_2 = 0$$

for all $(h_1, h_2) \in \tau_0(u_0(t))$ $(t \in T)$ has nonnull solutions and therefore (x_0, u_0) is not τ_0-regular. On the other hand, $z \equiv 0$ is the only solution to the system

$$\dot{z}(t) = 0, \quad z(t)h_2 \leq 0$$

for all $(h_1, h_2) \in \tau_2(u_0(t))$ $(t \in T)$ since both $(0, -1)$ and $(1, 1)$ belong to $\tau_2(u_0(t))$ implying that $-z(t) \leq 0$ and $z(t) \leq 0$ $(t \in T)$, and so (x_0, u_0) is τ_2-regular. Let $\mu = (\mu_1, \mu_2) \equiv (0, 1)$ and $p \equiv 1$ so that

i. $\mu_\alpha(t) \geq 0$ and $\mu_\alpha(t)\varphi_\alpha(u_0(t)) = 0$ $(\alpha = 1, 2,\ t \in T)$,

ii. $\dot{p}(t) = 0 = -H_x(\tilde{x}_0(t), p(t), \mu(t), 1)$ $(t \in T)$,

iii. $H_u(\tilde{x}_0(t), p(t), \mu(t), 1) =$

$$(\mu_1(t) + \mu_2(t) - 1, p(t) - \mu_2(t)) = (0, 0)\ (t \in T)$$

and so $(x_0, u_0, p, \mu) \in \mathcal{E}$ with (x_0, u_0) a weakly normal solution to the problem. Note also that

$$\tau_1(u_0(t), \mu(t)) =$$

$$\{(h_1, h_2) \in \mathbf{R}^2 \mid -h_1 \leq 0,\ h_1 = h_2\}$$

and therefore the system

$$\dot{z}(t) = 0,$$

$$z(t)h_2 \leq 0 \text{ for all } (h_1, h_2) \in \tau_1(u_0(t), \mu(t))\ (t \in T)$$

has nonnull solutions implying that (x_0, u_0, μ) is not τ_1-regular.

Now, if we set $v = (v_1, v_2) \equiv (1, 0)$ and $y \equiv 0$, then $v(t) \in \tau_2(u_0(t))$, (y, v) solves $\dot{y}(t) = v_2(t)$ $(t \in T)$ together with $y(0) = y(1) = 0$, and so it satisfies (i). However,

$$J((x_0, u_0, p, \mu); (y, v)) = -2 < 0. \blacksquare$$

Our next example yields a stronger conclusion than the previous one. We shall exhibit a weakly normal solution (x_0, u_0) to (P) with (p, μ) a pair such that $(x_0, u_0, p, \mu) \in \mathcal{E}$, but $J((x_0, u_0, p, \mu); (y, v)) < 0$ for some $(y, v) \in Z$ satisfying (i) and $v(t) \in \tau_1(u_0(t), \mu(t))$ $(t \in T)$.

Example 4.2 Consider the problem (P) of minimizing

$$I(x, u) = \int_0^1 \{u_2(t) + u_3(t)\}dt$$

subject to

$$\dot{x}(t) = u_1^2(t) + u_2(t) - u_3(t) \quad (t \in [0, 1]),$$

$$u_1(t) \geq 0,\ u_2(t) \geq 0,\ u_3(t) \geq 0\ (t \in [0, 1])$$

and $x(0) = x(1) = 0$.

In this case we have $T = [0, 1]$, $n = 1$, $m = r = q = 3$, $\xi_0 = \xi_1 = 0$ and, for any $t \in T$, $x \in \mathbf{R}$, and $u \in \mathbf{R}^3$ with $u = (u_1, u_2, u_3)$,

$$L(t, x, u) = u_2 + u_3, \quad f(t, x, u) = u_1^2 + u_2 - u_3,$$

$$\varphi_1(u) = -u_1, \quad \varphi_2(u) = -u_2, \quad \varphi_3(u) = -u_3.$$

Observe first that

$$H(t, x, u, p, \mu, 1) =$$

$$p(u_1^2 + u_2 - u_3) - u_2 - u_3 + \mu_1 u_1 + \mu_2 u_2 + \mu_3 u_3$$

so that

$$H_u(t, x, u, p, \mu, 1) =$$

$$(2pu_1 + \mu_1, p - 1 + \mu_2, -p - 1 + \mu_3),$$

$$H_{uu}(t,x,u,p,\mu,1) = \begin{pmatrix} 2p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, for any $(x,u,p,\mu) \in Z \times X \times \mathcal{U}_3$ and $(y,v) \in Z$,

$$J((x,u,p,\mu);(y,v)) = -\int_0^1 2p(t)v_1^2(t)dt.$$

Note that

$$f_x(t,x,u) = 0, \quad f_u(t,x,u) = (2u_1, 1, -1),$$

and we have $\varphi_1'(u) = (-1,0,0)$, $\varphi_2'(u) = (0,-1,0)$, $\varphi_3'(u) = (0,0,-1)$.

Clearly $(x_0,u_0) \equiv (0,0)$ solves the problem and we have $I_a(u_0(t)) = \{1,2,3\}$. Thus

$$\tau_0(u_0(t)) =$$
$$\{h \in \mathbf{R}^3 \mid -h_1 = 0,\ -h_2 = 0,\ -h_3 = 0\},$$
$$\tau_2(u_0(t)) =$$
$$\{h \in \mathbf{R}^3 \mid -h_1 \le 0,\ -h_2 \le 0,\ -h_3 \le 0\}.$$

Since $f_x(\tilde{x}_0(t)) = 0$ and $f_u(\tilde{x}_0(t)) = (0,1,-1)$, the system

$$\dot{z}(t) = -A^*(t)z(t) = 0,$$
$$z^*(t)B(t)h = z(t)(h_2 - h_3) = 0$$

for all $(h_1,h_2,h_3) \in \tau_0(u_0(t))$ $(t \in T)$ has nonnull solutions and therefore (x_0,u_0) is not τ_0-regular. Note that $z \equiv 0$ is the only solution to the system

$$\dot{z}(t) = 0, \quad z(t)(h_2 - h_3) \le 0$$

for all $(h_1,h_2,h_3) \in \tau_2(u_0(t))$ $(t \in T)$ since both $(0,1,0)$ and $(0,0,1)$ belong to $\tau_2(u_0(t))$ implying that $z(t) \le 0$ and $-z(t) \le 0$ $(t \in T)$, and so (x_0,u_0) is τ_2-regular. Now, let $\mu = (\mu_1,\mu_2,\mu_3) \equiv (0,0,2)$ and $p \equiv 1$. Then

$$\tau_1(u_0(t),\mu(t)) =$$
$$\{h \in \mathbf{R}^3 \mid -h_1 \le 0,\ -h_2 \le 0,\ -h_3 = 0\}$$

and therefore the system

$$\dot{z}(t) = 0, \quad z(t)(h_2 - h_3) \le 0$$

for all $(h_1,h_2,h_3) \in \tau_1(u_0(t),\mu(t))$ $(t \in T)$ has nonnull solutions implying that (x_0,u_0,μ) is not τ_1-regular. Now, we have

i. $\mu_\alpha(t) \ge 0$ and $\mu_\alpha(t)\varphi_\alpha(u_0(t)) = 0$ $(\alpha = 1,2,\ t \in T)$;
ii. $\dot{p}(t) = 0 = -H_x(\tilde{x}_0(t),p(t),\mu(t),1)$ $(t \in T)$;
iii. $H_u(\tilde{x}_0(t),p(t),\mu(t),1) = (0,0,0)$ $(t \in T)$,

so that $(x_0,u_0,p,\mu) \in \mathcal{E}$ with (x_0,u_0) a weakly normal solution to the problem.

Let $v = (v_1,v_2,v_3) \equiv (1,0,0)$ and $y \equiv 0$. Then (y,v) solves $\dot{y}(t) = v_2(t) - v_3(t)$ $(t \in T)$ together with $y(0) = y(1) = 0$, $v(t) \in \tau_1(u_0(t),\mu(t))$, and

$$J((x_0,u_0,p,\mu);(y,v)) = -2 < 0. \blacksquare$$

The following example yields a conclusion even stronger than those of the previous two examples. We shall exhibit a weakly normal solution (x_0,u_0) to (P) with (p,μ) a pair such that $(x_0,u_0,p,\mu) \in \mathcal{E}$, but $J((x_0,u_0,p,\mu);(y,v)) < 0$ for some $(y,v) \in Z$ satisfying (i) and $v(t) \in \tau_0(u_0(t))$ $(t \in T)$. In other words, the conclusion of Theorem 2.5 may not hold if we assume that the solution to the problem is weakly normal.

Example 4.3 Consider the problem of minimizing

$$I(x,u) = \int_0^1 u_2(t)dt$$

subject to

$$\dot{x}(t) = u_1^2(t) + u_2(t) - u_3(t)\ (t \in [0,1]),$$
$$u_2(t) \ge 0, \quad u_3(t) \ge 0 \quad (t \in [0,1]),$$

and $x(0) = x(1) = 0$.

In this case $T = [0,1]$, $n = 1$, $m = 3$, $r = q = 2$, $\xi_0 = \xi_1 = 0$ and, for all $t \in T$, $x \in \mathbf{R}$, and $u = (u_1,u_2,u_3)$,

$$L(t,x,u) = u_2, \quad f(t,x,u) = u_1^2 + u_2 - u_3,$$
$$\varphi_1(u) = -u_2, \quad \varphi_2(u) = -u_3.$$

We have

$$H(t,x,u,p,\mu) =$$
$$p(u_1^2 + u_2 - u_3) - u_2 + \mu_1 u_2 + \mu_2 u_3$$

and so

$$H_u(t,x,u,p,\mu) = (2pu_1, p - 1 + \mu_1, -p + \mu_2),$$
$$H_{uu}(t,x,u,p,\mu) = \begin{pmatrix} 2p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that, for all $(x,u,p,\mu) \in Z \times X \times \mathcal{U}_3$ and $(y,v) \in Z$,

$$J((x,u,p,\mu);(y,v)) = -\int_0^1 2p(t)v_1^2(t)dt.$$

Clearly $(x_0,u_0) \equiv (0,0)$ is a solution to the problem. Let $\mu = (\mu_1,\mu_2) \equiv (0,1)$ and $p \equiv 1$. Note that $(x_0, u_0, p, \mu) \in \mathcal{E}$. Now, $\varphi_1'(u) = (0, -1, 0)$, $\varphi_2'(u) = (0, 0, -1)$. Therefore

$$\tau_0(u_0(t)) = \{h \in \mathbf{R}^3 \mid -h_2 = 0,\ -h_3 = 0\},$$

$$\tau_1(u_0(t), \mu(t)) = \{h \in \mathbf{R}^3 \mid -h_2 \le 0,\ -h_3 = 0\},$$

$$\tau_2(u_0(t)) = \{h \in \mathbf{R}^3 \mid -h_2 \le 0,\ -h_3 \le 0\}.$$

Since

$$f_x(t, x_0(t), u_0(t)) = 0,$$

$$f_u(t, x_0(t), u_0(t)) = (0, 1, -1),$$

the system

$$\dot{z}(t) = -A^*(t)z(t) = 0,$$

$$z^*(t)B(t)h = z(t)(h_2 - h_3) = 0$$

for all $(h_1, h_2, h_3) \in \tau_0(u_0(t))$ $(t \in T)$ has nontrivial solutions and so (x_0, u_0) is not τ_0-regular. On the other hand, $z \equiv 0$ is the only solution to the system

$$\dot{z}(t) = 0,\ z(t)(h_2 - h_3) \le 0$$

for all $(h_1, h_2, h_3) \in \tau_2(u_0(t))$ $(t \in T)$ since both $(0, 1, 0)$ and $(0, 0, 1)$ belong to $\tau_2(u_0(t))$ implying that $z(t) \le 0$ and $-z(t) \le 0$ $(t \in T)$, so that (x_0, u_0) is τ_2-regular. Finally, the system

$$\dot{z}(t) = 0,\ z(t)(h_2 - h_3) \le 0$$

for all $(h_1, h_2, h_3) \in \tau_1(u_0(t), \mu(t))$ $(t \in T)$ has nontrivial solutions implying that (x_0, u_0, μ) is not τ_1-regular.

Let $v = (v_1, v_2, v_3) \equiv (1, 0, 0)$ and $y \equiv 0$. Then (y, v) solves $\dot{y}(t) = v_2(t) - v_3(t)$ $(t \in T)$, $y(0) = y(1) = 0$, $v(t) \in \tau_0(u_0(t))$, and

$$J((x_0, u_0, p, \mu); (y, v)) = -2 < 0. \blacksquare$$

The three previous examples deal with weakly normal solutions which yield negative second variations on $\tau_2(u_0(t))$, $\tau_1(u_0(t), \mu(t))$ and $\tau_0(u_0(t))$ respectively. We end with a fourth example showing that a τ_1-regular solution may yield a negative second variation on $\tau_2(u_0(t))$.

Example 4.4 Consider the problem of minimizing

$$I(x, u) = \int_0^1 \{-\exp(-u_3(t))\}dt$$

subject to

$$\dot{x}(t) = u_3(t) - u_1(t) + u_1^2(t)u_2(t) + 2\ (t \in [0, 1]),$$

$$u_1(t) - u_3(t) + u_1^8(t)u_2(t) \le 1,$$

$$u_3(t) - u_1(t) \le -2, \quad -u_3(t) \le 1 \quad (t \in [0, 1]),$$

and $x(0) = 0$, $x(1) = -1$.

In this case $T = [0, 1]$, $n = 1$, $m = q = r = 3$, $\xi_0 = 0$, $\xi_1 = -1$ and, for all $t \in T$, $x \in \mathbf{R}$, and $u = (u_1, u_2, u_3)$,

$$L(t, x, u) = -\exp(-u_3),$$

$$f(t, x, u) = u_3 - u_1 + u_1^2 u_2 + 2,$$

$$\varphi_1(u) = -1 + u_1 - u_3 + u_1^8 u_2,$$

$$\varphi_2(u) = 2 + u_3 - u_1, \quad \varphi_3(u) = -1 - u_3.$$

We have

$$H(t, x, u, p, \mu) = p(u_3 - u_1 + u_1^2 u_2 + 2) +$$

$$\exp(-u_3) - \mu_1(-1 + u_1 - u_3 + u_1^8 u_2) -$$

$$\mu_2(2 + u_3 - u_1) - \mu_3(-1 - u_3)$$

and so $H_x(t, x, u, p, \mu) = 0$ and

$$H_u(t, x, u, p, \mu) =$$

$$(p(-1 + 2u_1u_2) - \mu_1(1 + 8u_1^7 u_2) + \mu_2,$$

$$pu_1^2 - \mu_1 u_1^8, p - \exp(-u_3) + \mu_1 - \mu_2 + \mu_3).$$

Clearly $(x_0, u_0) \equiv (-t, 1, -1, -1)$ is a solution to the problem. Let $p \equiv 0$ and set $\mu = (\mu_1, \mu_2, \mu_3) \equiv (0, 0, e)$. Note that $(x_0, u_0, p, \mu) \in \mathcal{E}$. Now,

$$\varphi_1'(u) = (1 + 8u_1^7 u_2, u_1^8, -1),$$

$$\varphi_2'(u) = (-1, 0, 1), \quad \varphi_3'(u) = (0, 0, -1).$$

Therefore

$$\tau_0(u_0(t)) = \{h \in \mathbf{R}^3 \mid -7h_1 + h_2 - h_3 = 0,$$

$$-h_1 + h_3 = 0,\ -h_3 = 0\},$$

$$\tau_1(u_0(t), \mu(t)) = \{h \in \mathbf{R}^3 \mid -7h_1 + h_2 - h_3 \le 0,$$

$$-h_1 + h_3 \le 0,\ -h_3 = 0\},$$

$$\tau_2(u_0(t)) = \{h \in \mathbf{R}^3 \mid -7h_1 + h_2 - h_3 \le 0,$$

$$-h_1 + h_3 \le 0,\ -h_3 \le 0\}.$$

Since

$$f_x(t, x_0(t), u_0(t)) = 0,$$

$$f_u(t, x_0(t), u_0(t)) = (-3, 1, 1),$$

z solves the system

$$\dot{z}(t) = -A(t)z(t) = 0, \quad z(t)B(t)h \le 0$$

for all $h \in \tau_1(u_0(t), \mu(t))$ $(t \in T)$ if and only if

$$z \equiv c, \quad c(-3h_1 + h_2 + h_3) \le 0,$$

$$-7h_1 + h_2 - h_3 \le 0,$$

$$-h_1 + h_3 \leq 0, \quad -h_3 = 0 \quad (t \in T).$$

Thus $z \equiv 0$ is the only solution to the system given above and hence (x_0, u_0, μ) is τ_1-regular.

Let $v_1 = v_3 \equiv 1$,

$$v_2(t) := \begin{cases} 3 & \text{if } t \in [0, 1/2] \\ 1 & \text{if } t \in (1/2, 1] \end{cases}$$

$$y(t) := \begin{cases} t & \text{if } t \in [0, 1/2] \\ 1 - t & \text{if } t \in [1/2, 1] \end{cases}$$

Set $v = (v_1, v_2, v_3)$ and note that (y, v) solves

$$\dot{y}(t) = -3v_1(t) + v_2(t) + v_3(t) \ (t \in T),$$

$y(0) = y(1) = 0$ and $v(t) \in \tau_2(u_0(t))$. Moreover,

$$J((x_0, u_0, p, \mu); (y, v)) = \int_0^1 -ev_3^2(t)dt = -e < 0. \blacksquare$$

IV. Conclusions

In this paper we pose certain questions related to the nonnegativity of a quadratic form which occurs in optimal control problems with equality and/or inequality constraints in the control functions. In particular, the standard assumption of strong normality and other weaker assumptions are characterized in terms of certain convex cones.

These characterizations allow us to provide several examples of solutions to optimal control problems which yield negative second variations on some of those sets which form different critical directions. In all the examples, the solutions fail to be strongly normal but satisfy first order conditions with a positive cost multiplier together with some regularity assumptions. It is of interest to see if, in all cases, the strong normality assumption is fundamental in the derivation of second order conditions of this kind, and this paper illustrates the behaviour in some problems of regular solutions that fail to satisfy that assumption.

References

[1] A.V. Arutyunov and F.L. Pereira, Second-order necessary optimality conditions for problems without a priori normality assumptions, *Mathematics of Operations Research* 31, 2006, pp. 1-12.

[2] A.V. Arutyunov and Y.S. Vereshchagina, On necessary second-order conditions in optimal control problems (Russian) *Differentsial'nye Uravneniya* 30, 2002, pp. 1443-1450; translation in *Differential Equations* 38, 2002, pp. 1531-1540.

[3] H.A. Biswas, Necessary conditions for optimal control problems with and without state constraints: a comparative study, *WSEAS Transactions on Systems and Control* 6, 2011, pp. 217-228.

[4] M.R. de Pinho and J.F. Rosenblueth, Mixed constraints in optimal control: an implicit function theorem approach, *IMA Journal of Mathematical Control and Information* 24, 2007, pp. 197–218.

[5] E.G. Gilbert and D.S. Bernstein, Second order necessary conditions in optimal control: accessory-problem results without normality conditions, *Journal of Optimization Theory & Applications* 41, 1983, pp. 75-106.

[6] M.R. Hestenes, *Calculus of Variations and Optimal Control Theory*, John Wiley & Sons, New York, 1966.

[7] M.R. Hestenes, *Optimization Theory, The Finite Dimensional Case*, John Wiley & Sons, New York, 1975.

[8] E. Levitin, A. Milyutin, and N.P. Osomolovskiĭ, Conditions of high order for a local minimum for problems with constraints, *Russian Math Surveys* 33, 1978, pp. 97-168.

[9] P.D. Loewen and H. Zheng, Generalized conjugate points for optimal control problems, *Nonlinear Analysis, Theory, Methods & Applications* 22, 1994, pp. 771–791.

[10] P.D. Loewen and H. Zheng, Generalized conjugate points in optimal control, *Proceedings of the 33rd IEEE Conference on Decision and Control*, Lake Buena Vista, Florida, 4, 1994, pp. 4004–4008.

[11] H. Maurer and S. Pickenhain, Second order sufficient conditions for control problems with mixed control-state constraints, *Journal of Optimization Theory and Applications* 86, 1995, pp. 649–667.

[12] H. Maurer and H.J. Oberle, Second order sufficient conditions for optimal control problems with free final time: the Riccati approach, *SIAM Journal on Control and Optimization* 41, 2002, pp. 380–403.

[13] A.A. Milyutin and N.P. Osmolovskiĭ, *Calculus of Variations and Optimal Control*, Translations of Mathematical Monographs, 180, American Mathematical Society, Providence, Rhode Island, 1998.

[14] B.S. Mordukhovich, Complete characterization of openness, metric regularity, and Lipshitzian properties of multifunctions, *Transactions of the American Mathematical Society* 340, 1993, pp. 1-36.

[15] N.P. Osmolovskiĭ, Second order conditions for a weak local minimum in an optimal control problem (necessity, sufficiency), *Soviet Math Dokl* 16, 1975, pp. 1480-1484.

[16] I.B. Russak, Second order necessary conditions for problems with state inequality constraints, *SIAM Journal on Control* 13, 1975, pp. 372-388.

[17] I.B. Russak, Second order necessary conditions for general problems with state inequality constraints, *Journal of Optimization Theory and Applications* 17, 1975, pp. 43-92.

[18] J.F. Rosenblueth, Admissible variations for optimal control problems with mixed constraints, *WSEAS Transactions on Systems* 4, 2005, pp. 2204–2211.

[19] J.F. Rosenblueth, Augmented integrals for optimal control problems, *International Journal of Applied Mathematics and Informatics* (submitted)

[20] J.F. Rosenblueth and G. Sánchez Licea, Negative Second Variations for Problems with Inequality Control Constraints, *3rd European Conference for the Applied Mathematics and Informatics, Montreux, Suiza*, WSEAS Press, Mathematical Methods for Information Science and Economics, 2012, pp. 150-155.

[21] G. Stefani and P.L. Zezza, Optimality conditions for a constrained control problem, *SIAM Journal on Control & Optimization* 34, 1996, pp. 635-659.