Comparison of the accuracy of L-moments, TL-moments and maximum likelihood methods of parameter estimation

Diana Bílková

Abstract**—**Despite not producing good results, the method of moments is commonly applied when constructing the most appropriate parametric distribution for a given data file. An alternative approach is to use the so-called order statistics. The present paper deals with the application of order statistics (parameter estimation methods of L-moments and TL-moments) to the economic data. Theoretical advantages of L-moments over conventional moments become obvious when applied to small data sets, e.g. in hydrology, meteorology and climatology, considering extreme precipitation in particular. L-moments have been introduced as a robust alternative to classical moments of probability distributions. However, L-moments and their estimates lack some robust features specific to TL-moments, the latter representing an alternative robust version of the former, the so-called trimmed L-moments.

The main aim of this paper is to apply the two methods to large data sets, comparing their parametric estimation accuracy with that of the maximum likelihood method. In this very case, the methods of L-moments and TL-moments are utilized for the construction of income and wage distribution models. Three-parameter lognormal curves represent the basic theoretical probability distribution whose parameters were estimated simultaneously by the three methods of point parameter estimation, their accuracy having been then evaluated.

Income and wage distributions for the Czech Republic have been examined. The total of 168 nominal income distributions (net annual household income per capita in CZK) for the years 1992, 1996, 2002 (Microcensus survey) and 2004–2007 (EU Statistics on Income and Living Conditions survey) were analyzed, both the total income distribution for all Czech households and income distribution figures broken down into gender, historical land (Bohemia, Moravia), social group, municipality size, age and educational attainment having been studied. In addition, a total of 328 nominal wage distributions (gross monthly wage in CZK) have become the subject of the research; the total wage distribution for all CR employees as well as wage distributions in terms of gender, age, educational attainment and the classification of jobs and economic activities being examined. 2003–2010 data in the form of an interval frequency distribution were drawn from the official website of the Czech Statistical Office.

The study is divided into a theoretical part, in which mathematical and statistical aspects are described, and an analytical part, where the results of the three robust parameter estimation methods are presented. For all analyzed income and wage distributions, the model distribution parameters were estimated using the methods of TL-moments, L-moments and maximum likelihood simultaneously. The accuracy of the methods employed was then compared, TL-moments having brought the most accurate, L-moments the second best and the maximum likelihood method the least accurate results in general.

Keywords**—**Distribution function, L-moments and TL-moments, order statistics, quantile function.

Highlights**—**We use individual data on the net annual household income per capita and data in the form of an interval frequency distribution on the gross monthly wage in CZK.
We use three-parameter lognormal curves as a basic theoretical probability distribution.
We use the methods of L-moments and TL-moments and the maximum likelihood method.
The method of TL-moments has brought the most accurate, the method of L-moments the second most accurate and the maximum likelihood method the least accurate results.

Mathematics Subject Classification**—**60E05, 62E99, 62H12, 62F10.

I. Introduction

THE advantages of L-moments and TL-moments methods are obvious when applied to small data sets, predominantly in the fields of hydrology, meteorology and climatology, considering extreme precipitation in particular. The main aim of this paper is to utilize the two methods of parameter estimation in large data sets and compare their accuracy to that of the maximum likelihood method.

The total income distribution for all Czech households as well as income distributions broken down by gender, historical land (Bohemia, Moravia), social group, municipality size, age and educational attainment are examined. The total wage distribution for all employees of the Czech Republic and wage distributions divided in terms of gender, job category, economic activity, age and educational attainment are also studied. Altogether, 168 distributions of net annual household income in CZK (nominal income) and 328 distributions of gross monthly wage in CZK (nominal wage) have been researched. The income distribution data coming from the years 1992, 1996 and 2002 originate from the Microcensus statistical survey, those for the years 2004–2007 being generated from the EU-SILC survey. All the data were taken from the Czech Statistical Office. Data concerning the wage

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Diana Bílková is with the Department of Statistics and Probability, Faculty of Informatics and Statistics, University of Economics, Prague, Sq. W. Churchill 1938/4, 130 67 Prague 3, Czech Republic (e-mail: bilkova@vse.cz).

distribution (in the form of an interval frequency distribution) for the period 2003–2010 were also downloaded from the official CSO website. Three-parameter lognormal curves represent the basic theoretical probability distribution. For all analyzed income distributions, the model distribution parameters were estimated using the methods of L-moments, TL-moments and maximum likelihood simultaneously, their accuracy having been subsequently compared.

L-moments form the basis for a general theory that includes the summarization and description of theoretical probability distributions and obtained sample data sets, parameter estimation of theoretical probability distributions and hypothesis testing of their parameter values. The theory of L-moments includes such established methods like the use of order statistics and the Gini middle difference. It leads to some promising innovations in the area of measuring skewness and kurtosis of the distribution, providing relatively new methods of parameter estimation for a particular distribution. L-moments can be defined for any random variable whose expected value exists. The main advantage of L-moments over conventional moments is that the former can be estimated by linear functions of sample values and are more resistant to the influence of sample variability. L-moments are more robust than conventional moments to the existence of outliers in the data, allowing better conclusions reached on the basis of small samples of the basic probability distribution. L-moments sometimes bring even more efficient parameter estimations of parametric distribution than the estimations acquired using the maximum likelihood method, particularly for small samples.

L-moments are an alternative system describing the shape of the probability distribution. They have certain theoretical advantages over conventional moments resting in the ability to characterize a wider range of distribution. They are more resistant to outliers compared with conventional moments and less prone to estimation bias, the approximation by asymptotic normal distribution being more accurate in finite samples. L-moments are analogous to conventional moments. They can be estimated based on linear combinations of sample order statistics, i.e. L-statistics. L-moments and their estimations, however, lack some robust features that belong to TL-moments, the latter (the trimmed L-moments) representing an alternative robust version of the former.

All calculations were performed using Statgraphics and SAS statistical software packages, the Microsoft Excel spreadsheet and mathematical software R.

II. Material and Methods

A. Wage Distribution

The research database consists of the total wage distribution for all employees in the Czech Republic together and the wage distribution of relatively homogeneous groups of the Czech population broken down by gender, job classification (see Table I), industrial classification of economic activities (Tables II and III), age and educational attainment (Table IV); all distributions were measured in the period 2003–2010. In the observed years, however, a substantial change occurred in the economic activity nomenclature, the standard industrial classification (SIC) having been replaced by the new statistical nomenclature (CZ-NACE); see Tables II and III. This disrupts the continuity of the time series obtained over the given period, data for the years 2003–2008 being based on the former while those for the 2009–2010 period on the latter classification.

The research design involved the employees in the Czech Republic. The gross monthly wage in CZK (nominal wage) was the research variable, interval frequency distributions with extreme open intervals being the subject of the research. They are indicated in the following standard tables – available in the Czech Statistical Office website –, presenting percentages of employees in gross monthly wage brackets broken down by:

- gender;
- main job categories;
- industry;
- age;
- educational attainment.

The following CSO analytical tables provided details of the survey sample (sample sizes):

- number of employees and their average gross monthly wages in the main job categories according to educational attainment – males;
- number of employees and their average gross monthly wages in the main job categories according to educational attainment – females;
- number of employees and their average gross monthly wages in the main job categories according to educational attainment;
- number of employees and their average gross monthly wages according to industry and age;
- number of employees and their average gross monthly wages according to age and educational attainment.

Additional data were adopted from the following regional table on the CSO website:

- structure of the average gross monthly wage in the regions.

Tables V–X give information on sample sizes of the above wage distributions.

The Czech Statistical Office (CSO) draws information on the development of gross monthly wages from two sources simultaneously. Enterprise payroll reporting is the first one, offering reliable data on wages in the national economy that can be sorted by business criteria, such as sectors or size groups, not providing more detailed classification though. Structural statistics are another source of data. They provide the most detailed information on wages of individual employees using various classification criteria, occupational ones in particular. The Czech Statistical Office receives an overview of the wage distribution among employees from those statistics as well.

Table I Job classification

Main job categories	Code
Legislative, leading and managing staff	1,000
Scientific and professional staff	2,000
Technical, medical and pedagogical staff	3,000
Junior administrative staff	4,000
Operational staff in services and trade	5,000
Qualified workers in agriculture, forestry and fishing	6,000
Craftsmen, qualified producers and processors	7,000
Machine and equipment operators	8,000
Auxiliary and unqualified workers	9,000

Source: http://www.czso.cz

Table II Industrial classification of economic activities (2003–2008)

Sections of industrial classification of economic activities (SIC)	Marker
Agriculture, forestry and fishing	A+B
Industry	C-E
Building industry	F
Trade; maintenance of motor vehicles and products for personal and household consumption	G
Accommodation and catering	H
Transportation, warehousing and communications	I
Financial intermediation	J
Real estate and rental activities; entrepreneurial activities	K
Public administration and defense; compulsory social security	L
Education	M
Health and social care; veterinary activities	N
Other public, social and personal services	O

Source: http://www.czso.cz

Table III Classification of economic activities (2009–2010)

Nomenclature of economic activities (CZ-NACE)	Marker
Agriculture, forestry and fishing	A
Industry	B-E
Building industry	F
Trade; maintenance of motor vehicles	G
Transportation and warehousing	H
Accommodation, catering and hospitality	I
Information and communication activities	J
Finance and insurance	K
Real estate activities	L
Professional, scientific and technical activities	M
Administrative and support activities	N
Public administration and defense, compulsory social security	O
Education	P
Health and social care	Q
Cultural, entertainment and recreational activities	R
Other activities	S

Source: http://www.czso.cz

Table IV Classification by age and educational attainment

Age	Education
to 19 years	Primary and incomplete
from 20 to 24 years	Secondary without high school diploma
from 25 to 29 years	Secondary with high school diploma
from 30 to 34 years	Higher professional and undergraduate
from 35 to 39 years	Tertiary (2nd degree)
from 40 to 44 years	
from 45 to 49 years	
from 50 to 54 years	
from 55 to 59 years	
from 60 to 64 years	
from 65 years	

Source: http://www.czso.cz

The CSO has cooperated with the Ministry of Labor and Social Affairs in terms of structural statistics since 1996, finding out about individual employees' wages. Thus, apart from gross wage components, individual employees' personal data, such as gender, age and educational attainment are under scrutiny. The collected statistics data are used for a detailed analysis of the labor market and its development. In structural statistics on gross wages, all wages earned for work performed, including premiums, bonuses or additional salaries as well as earning compensations for time not worked during vacations or due to work impediments are reported. The average wage of an employee in a given year is calculated in relation to his/her paid time, i.e. the number of months he/she really receives a wage or its compensation. The duration of sickness and other unpaid absences from work is therefore not included. The calculated average gross monthly wage precisely characterizes comparable wage levels of different jobs, being based on an exactly given amount of paid time. The average gross wage calculated in this way is not the same as that obtained from standard business reports that measure the total volume of wages against the number of registered staff of an organization, including those on sickness or other unpaid leave of less than four weeks. Except for the effect of unpaid leaves and a different database, further differences between the wage levels relative to other statistical sources may arise due to the fact that employees whose weekly working load is less than thirty hours are not included in the structural statistics. Results of structural statistics are produced by the sample survey, being therefore affected by a sample error. Moreover, some of the addressed units do not provide the data required for the sample analysis and certain records have to be excluded because of a high error rate, which causes minor distortions; see more at www.czso.cz.

Table V Sizes of sample sets of wage distribution broken down by gender

	Year							
Gender	2003	2004	2005	2006	2007	2008	2009	2010
Total	1,018,934	1,404,496	1,515,527	1,614,372	1,673,498	1,711,811	1,651,506	1,662,829
Male	559,863	711,551	769,802	813,821	858,656	875,139	846,028	850,788
Female	459,071	692,945	745,725	800,551	814,842	836,672	805,478	812,041

Source: http://www.czso.cz

Table VI Sizes of sample sets of wage distribution broken down by job classification

	Year							
Code	2003	2004	2005	2006	2007	2008	2009	2010
1000	60,300	84,264	91,302	96,382	104,516	107,599	109,281	110,155
2000	109,779	241,959	248,320	270,252	273,497	285,880	289,894	295,775
3000	250,639	355,319	383,730	402,651	402,553	413,067	399,798	401,402
4000	77,565	95,552	101,920	111,470	118,124	122,083	123,784	125,778
5000	63,685	95,247	108,172	122,661	128,053	134,127	134,560	134,370
6000	9,912	10,697	11,417	10,098	8,859	7,877	7,630	7,250
7000	193,715	211,356	226,527	232,399	243,246	243,390	221,308	225,420
8000	192,378	214,229	240,057	258,177	282,001	284,634	260,355	256,472
9000	60,961	95,873	104,082	110,282	112,649	113,154	104,896	106,207

Source: http://www.czso.cz

Table VII Sizes of sample sets of wage distribution broken down by the industrial classification of economic activities

	Year							
Marking	2003	2004	2005	2006	2007	2008	2009	2010
A+B	28,132	31,055	33,004	27,502	24,296	21,537	−	−
C-E	431,534	479,817	522,097	554,783	600,924	603,951	−	−
F	38,261	42,223	45,242	43,941	50,073	50,437	−	−
G	52,070	63,221	74,232	93,353	111,944	120,464	−	−
H	8,556	11,188	12,020	15,447	16,858	16,997	−	−
I	161,895	157,881	142,185	141,819	143,612	144,536	−	−
J	47,932	52,140	48,601	51,893	53,506	55,993	−	−
K	35,911	43,758	49,080	59,836	67,604	79,003	−	−
L	68,971	192,993	217,590	235,536	232,800	233,438	−	−
M	33,508	173,477	183,277	189,068	187,325	188,730	−	−
N	93,480	125,784	149,429	160,700	144,471	155,533	−	−
O	18,684	30,959	38,770	40,494	40,085	41,192	−	−

Source: http://www.czso.cz

Table VIII Sizes of sample sets of wage distribution broken down by the NACE classification of economic activities

	Year							
Marking	2003	2004	2005	2006	2007	2008	2009	2010
A	−	−	−	−	−	−	20,560	18,659
B-E	−	−	−	−	−	−	558,904	560,299
F	−	−	−	−	−	−	50,789	52,769
G	−	−	−	−	−	−	125,373	130,348
H	−	−	−	−	−	−	147,328	141,193
I	−	−	−	−	−	−	17,132	16,673
J	−	−	−	−	−	−	42,058	43,602
K	−	−	−	−	−	−	57,149	57,715
L	−	−	−	−	−	−	5,540	5,093
M	−	−	−	−	−	−	20,922	22,978
N	−	−	−	−	−	−	41,588	44,533
O	−	−	−	−	−	−	208,606	212,765
P	−	−	−	−	−	−	185,453	186,092
Q	−	−	−	−	−	−	143,595	143,877
R	−	−	−	−	−	−	23,756	23,033
S	−	−	−	−	−	−	2,753	3,200

Source: http://www.czso.cz

Table IX Sizes of sample sets of wage distribution broken down by age

Age (in years)	Year 2003	2004	2005	2006	2007	2008	2009	2010
− 19	2,805	3,567	4,314	5,887	6,879	6,455	4,245	3,927
20 – 24	63,496	76,595	86,317	97,025	105,523	106,958	94,097	91,160
25 – 29	129,298	166,682	178,259	188,289	193,222	190,866	177,961	177,044
30 – 34	121,054	173,799	197,020	217,720	227,325	231,284	220,500	216,899
35 − 39	122,324	170,268	183,513	198,609	210,780	226,740	233,095	246,619
40 − 44	123,278	184,904	204,368	218,373	225,528	226,265	216,461	218,695
45 − 49	148,936	198,188	205,107	208,653	209,454	217,468	220,087	227,237
50 − 54	166,456	221,988	222,759	220,744	220,894	216,944	201,687	194,387
55 − 59	113,813	163,222	182,059	194,592	200,682	207,352	201,606	203,674
60 − 64	22,019	36,571	42,151	52,473	60,501	66,795	66,452	68,220
65 +	5,455	8,712	9,660	12,007	12,710	14,684	15,315	14,967

Source: http://www.czso.cz

Table X Sizes of sample sets of wage distribution broken down by educational attainment

Education	Year 2003	2004	2005	2006	2007	2008	2009	2010
Primary and incomplete	95,112	119,480	125,972	129,027	135,399	137,190	120,254	116,383
Secondary without high school diploma	377,347	470,688	523,744	553,522	587,081	591,669	557,780	555,266
Secondary with high school diploma	408,562	560,237	575,668	621,306	629,447	644,576	625,631	627,073
Higher profess. and undergraduate	15,749	29,144	40,055	42,856	47,967	54,439	57,747	64,684
Tertiary (2^{nd} deg.)	122,164	224,947	250,088	267,661	273,604	283,937	290,094	299,423

Source: http://www.czso.cz

B. *Income Distribution*

In terms of the accuracy of different methods of point parameter estimation, the results obtained by the analysis of wage distribution are compared to those produced by the research of income distribution in the Czech Republic. This project examined the total distribution of net annual household income per capita in CZK (nominal income) for all households of the Czech Republic together as well as income distribution figures broken down into gender, historical land (Bohemia and Moravia; see Fig. 1), social group (junior employee, self-employed, senior employee, pensioners either economically active or inactive, unemployed and others), municipality size (0–999, 1,000–9,999, 10,000–99,999, 100,000 and more inhabitants), age (0–29, 30–39, 40–49, 50–59, from 60 years of age) and educational attainment (primary, secondary, complete secondary, tertiary), households having been categorized according to the head of the household (male in the vast majority). Individual data for the years 1992 (sample of 16,233 households), 1996 (28,148), 2002 (7,973), 2004 (4,351), 2005 (7,483), 2006 (9,675) and 2007 (11,294) were collected by the Czech Statistical Office – the first three years based on the Microcensus survey findings, the latter four years on the EU-SILC statistical survey conducted between 2005 and 2008. The information on the sample sizes of these income distributions is presented in Table XI.

The head of the household is always a male in two-parent families (a husband-and-wife or cohabiting type), regardless of the economic activity. In single-parent families (a one-parent-with-children type) and non-family households whose members are related neither by marriage (partnership) nor parent-child relationship, the key criterion for determining the head of the household is economic activity, another aspect being the amount of money income of individual household members. The former criterion also applies in the case of more complex household types, for instance, in joint households of more two-parent families. For this reason, we can observe smaller sample sizes in the cases where the head of the household is a female as compared to those of male-headed households; see Table XI.

Different approaches to the parameter estimation and various types of estimators have been currently under discussion in some areas of statistics and applied mathematics, see, e.g. [13], [21], and [24]–[25], the application of the L-moments method of the parameter estimation being studied, e.g. in [2]−[11]. The development and modeling of wage and income distributions have been also dealt with extensively in the statistical literature; see, e.g. [19]–[20] or [22]–[23]. A secondary aim of the present paper is to analyze the development of wage and income distributions in the Czech Republic in the period prior to and, as for wage distribution, during the global economic downturn.

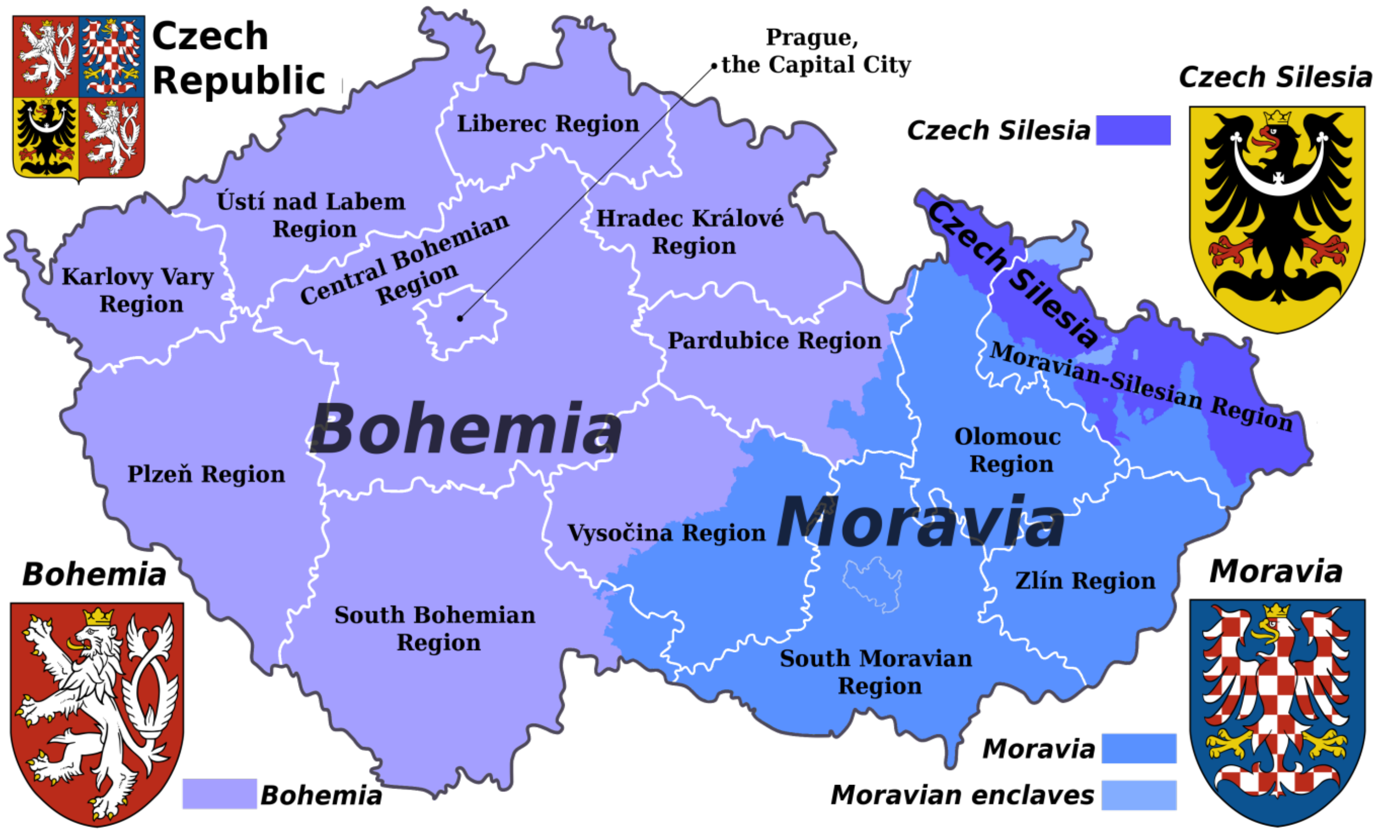

Fig. 1 Historical lands of the Czech Republic and their regions

Source: http://www.google.cz

Table XI Sample sizes of income distributions broken down by relatively homogeneous categories

		Year						
	Set	1992	1996	2002	2004	2005	2006	2007
Gender	Men	12,785	21,590	5,870	3,203	5,456	7,151	8,322
	Women	3,448	6,558	2,103	1,148	2,027	2,524	2,972
Country	Czech Republic	16,233	28,148	7,973	4,351	7,483	9,675	11,294
	Bohemia	9,923	22,684	5,520	2,775	4,692	6,086	7,074
	Moravia	6,310	5,464	2,453	1,576	2,791	3,589	4,220
Social group	Lower employee	4,953	4,963	1,912	1,068	1,880	2,385	2,811
	Self-employed	932	1,097	740	391	649	802	924
	Higher employee	3,975	4,248	2,170	1,080	1,768	2,279	2,627
	Pensioner with s EA	685	594	278	178	287	418	493
	Pensioner without EA	4,822	4,998	2,533	1,425	2,577	3,423	4,063
	Unemployed	189	135	172	131	222	258	251
Municip. size	0–999 inhabitants	2,458	3,069	999	727	1,164	1,607	1,947
	1,000–9,999 inhabitants	4,516	4,471	2,300	1,233	2,297	3,034	3,511
	10,000–99,999 inhabitants	5,574	5,755	2,401	1,508	2,655	3,347	3,947
	100,000 and more inhabitants	3,685	2,853	2,273	883	1,367	1,687	1,889
Age	To 29 years	1,680	2,809	817	413	627	649	827
	From 30 to 39 years	3,035	4,718	1,398	716	1,247	1,620	1,655
	From 40 to 49 years	3,829	6,348	1,446	738	1,249	1,609	1,863
	From 50 to 59 years	2,621	5,216	1,642	919	1,581	2,051	2,391
	From 60 years	5,068	9,057	2,670	1,565	2,779	3,746	4,558
Education	Primary	9,302	15,891	3,480	553	940	1,183	1,385
	Secondary	4,646	3,172	2,493	3,186	5,460	7,168	8,371
	Complete secondary	1,951	6,356	1,129	118	282	266	319
	Tertiary	334	2,729	871	494	801	1,058	1,219

Source: Own research

Let X be a random variable with the distribution function $F(x)$ and quantile function $x(F)$ and let $X_1, X_2, \ldots, X_n$ denote a random sample from the given distribution of the sample size n. Then $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ are the order statistics of the random sample of size n, coming from the distribution of the random variable X.

III. Theory and Calculation

A. Lognormal Distribution

The lognormal distribution was pioneered, for instance, by Galton, McAlister, Kapteyn, van Uven and Gibrat, the initial study of this probability distribution being followed up, e.g. by Fechner, Wicksell, Nydell, Davies, Yuan, Finney, Kalecki, Gaddum, Bliss, Hatch, Choute, Krumbein, Bol'shev, Prohorov, Rudinov, Herdan, Kalinske, Kolmogorov, Kottler, Wise, Cochran, Williams, Grundy, Herdan, Pearce, Koch, Aitchison, Brown, Wu and many others. Among more recent authors are, for example, Nakamura, Crow, Shimizu, Johnson, Balakrishnan, Kleiber or Kotz; see [16]–[17]. The lognormal distribution as a model for sample distributions is of unquestionable significance, its distinguishing properties being sequential actions of mutually dependent factors, a tendency towards the development in geometric progression and the conversion of a random to systematic variability, i.e. the differentiation. The lognormal model is applied in diverse fields such as astronomy, engineering or sociology.

In economics, wages and incomes of the population are among the many phenomena that the lognormal model allows to interpret. It is necessary to observe the following requirements.

- The curve must represent a given shape of the frequency distribution in the best possible way, being therefore most closely congruent with the respective modeled distribution in terms of its basic properties such as the location, variability, skewness and kurtosis.
- The shape of the curve is supposed to be as simple as possible so that it can be manipulated and, above all, it should depend on a small number of parameters that can be estimated by a suitable method of point estimation.
- Moreover, the interpretability of the curve parameters is required so that their values can be predicted without using the methods of a statistical time series analysis, particularly in the cases when sufficiently long time series are not available.

Every choice is always a certain compromise between the above mentioned requirements. Parameter functions of lognormal curves have a very simple interpretation. In the case of a two-parameter lognormal curve, the expression exp(μ) represents the median of the gross monthly wage or the median of net annual household income per capita, parameters μ and σ^2 representing the expected value and variance of natural logarithms of wages and incomes, respectively. In the case of a three-parameter lognormal curve, the parameter θ represents this curve's minimum, the expression exp(μ) indicating the distance of the wage or income median from this theoretical minimum. Parameters μ and σ^2 represent the expected value and variance of the natural logarithms of wage or income distances from the theoretical minimum θ.

The notion that the logarithms of economic variable values are normally distributed is slightly outdated. It stems from the fact that the effects of a large number of different impulses, resulting in the value of a monitored variable, are proportional to the present state of the variable.

However, a strong agreement of the model with global wage or income distributions does not imply that the lognormal distribution is appropriate for any case or, eventually, for extremely homogeneous employee or household subsets created through a detailed classification by certain demographic or socio-economic criteria.

As for the applicability of the lognormal distribution, it is obvious that the wage or income distribution can be captured by a lognormal curve with a sufficient accuracy in the standard case of not too detailed classification by relatively homogeneous subgroups of employees or households. The parameters of lognormal distribution can be properly estimated from the sample or, alternatively, the curve can be shifted either by subjectively determined wage or income minimum or by the shift parameter – the third of the parameters estimated on a sample basis. This solution brought about positive outcomes in the construction of global wage and income models, both on national scales and for relatively homogeneous large groups created only through a rough classification according to some demographic and socio-economic criteria. However, the lognormal model is not to be considered universal to such an extent that it would be suitable for any subset of employees or households created through a very detailed classification. (This kind of classification is not the subject of this research in terms of either wage or income distribution anyway.)

Two-Parameter Lognormal Distribution

The random variable X has a two-parameter lognormal distribution with parameters μ and σ^2, where $-\infty < \mu < \infty$, $\sigma^2 > 0$ if its probability density function has the form

$$f(x;\mu,\sigma^2) = \frac{1}{\sigma\cdot x\cdot\sqrt{2\pi}}\cdot\exp\left[-\frac{(\ln x-\mu)^2}{2\sigma^2}\right], \qquad x>0,$$

otherwise equaling 0. (1)

The lognormal distribution with parameters μ and σ^2 is denoted LN(μ, σ^2). The probability density function of the two-parameter lognormal distribution is asymmetric, positively skewed. Figs. 2 and 3 show a graph of the probability density function of the two-parameter lognormal distribution LN(μ, σ^2) depending on the values of distribution parameters μ and σ^2.

The probability density function of the two-parameter lognormal distribution is sometimes given in the form

$$f(x;\gamma,\delta) = \frac{\delta}{x\cdot\sqrt{2\pi}}\cdot\exp\left[-\frac{1}{2}(\gamma+\delta\cdot\ln x)^2\right], \qquad x>0,$$

otherwise equaling 0, (2)

probability density function formulae (1) and (2) being related as $\mu = -\frac{\gamma}{\delta}$ and $\sigma = \frac{1}{\delta}$.

We denote $\omega = \exp(\sigma^2)$, the r-th common and central moments of the two-parameter lognormal distribution having the form

$$\mu_r^{\backslash} = E(X^r) = \exp\left(r\cdot\mu + \frac{r^2\sigma^2}{2}\right), \tag{3}$$

$$\mu_r = E[(X-\mu_1^{\backslash})^r] = \omega^{r/2}\cdot\left[\sum_{j=0}^{r}(-1)^j\cdot\binom{r}{j}\cdot\omega^{(r-j)\cdot(r-j-1)/2}\right]\cdot\exp(r\cdot\mu), \tag{4}$$

specifically,

$$\mu_3 = \omega^{3/2}\cdot(\omega-1)^2\cdot(\omega+2)\cdot\exp(3\cdot\mu), \tag{5}$$

$$\mu_4 = \omega^2\cdot(\omega-1)^2\cdot(\omega^4+2\omega^3+3\omega^2-3)\cdot\exp(4\cdot\mu). \tag{6}$$

It follows from (3) and (4) that the expected value and variance of the random variable X, having a two-parameter lognormal distribution, depends on both parameters

$$E(X) = \exp\left(\mu + \frac{\sigma^2}{2}\right), \tag{7}$$

$$D(X) = \exp(2\mu+\sigma^2)\cdot[\exp(\sigma^2)-1] = \exp(2\mu)\cdot\omega\cdot(\omega-1). \tag{8}$$

On the other hand, the median (as well as the geometric mean $Geo(X)$ in this case) depends only on one parameter μ

$$Median(X) = \exp(\mu), \tag{9}$$

which follows from the formula for a $100\cdot P\%$ quantile of this distribution

$$x_P = \exp(\mu + \sigma\cdot u_P), \tag{10}$$

where u_P is the $100\cdot P\%$ quantile of the standardized normal distribution. Thus, it holds that

$$Median(X) = Geo(X) = \exp(\mu). \tag{11}$$

The two-parameter lognormal distribution is unimodal, with one mode

$$Mode(X) = \exp(\mu-\sigma^2) = \frac{\exp(\mu)}{\omega}. \tag{12}$$

The relation between the expected value, median and mode, resulting from (7), (9) and (12), which is typical especially for a positively skewed frequency distribution, is expressed as follows

$$E(X) > Median(X) > Mode(X). \tag{13}$$

The coefficient of variation of the two-parameter lognormal distribution, see [9], depends only on one variability parameter σ^2

$$V(X) = \sqrt{\exp(\sigma^2)-1} = \sqrt{\omega-1}. \tag{14}$$

Another interesting characteristic of the variability, the Gini coefficient, also depending – in the case of the two-parameter lognormal distribution – on a single parameter σ^2, has the form

$$G = \operatorname{erf}\left(\frac{\sigma}{2}\right), \tag{15}$$

alternatively,

$$G = 2\Phi\left(\frac{\sigma}{\sqrt{2}}\right) - 1, \tag{16}$$

see [9], where erf(z) is the so-called error function

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\cdot\int_0^z \exp(-t^2)\,\mathrm{d}t.$$

Moment measures of the skewness and kurtosis also depend on a single parameter σ^2

$$\beta_1 = \sqrt{\exp(\sigma^2)-1}\cdot[\exp(\sigma^2)+2] = \sqrt{\omega-1}\cdot(\omega+2), \tag{17}$$

$$\beta_2 = [\exp(4\sigma^2)+2\exp(3\sigma^2)+3\exp(2\sigma^2)-3] = (\omega^4+2\omega^3+3\omega^2-3). \tag{18}$$

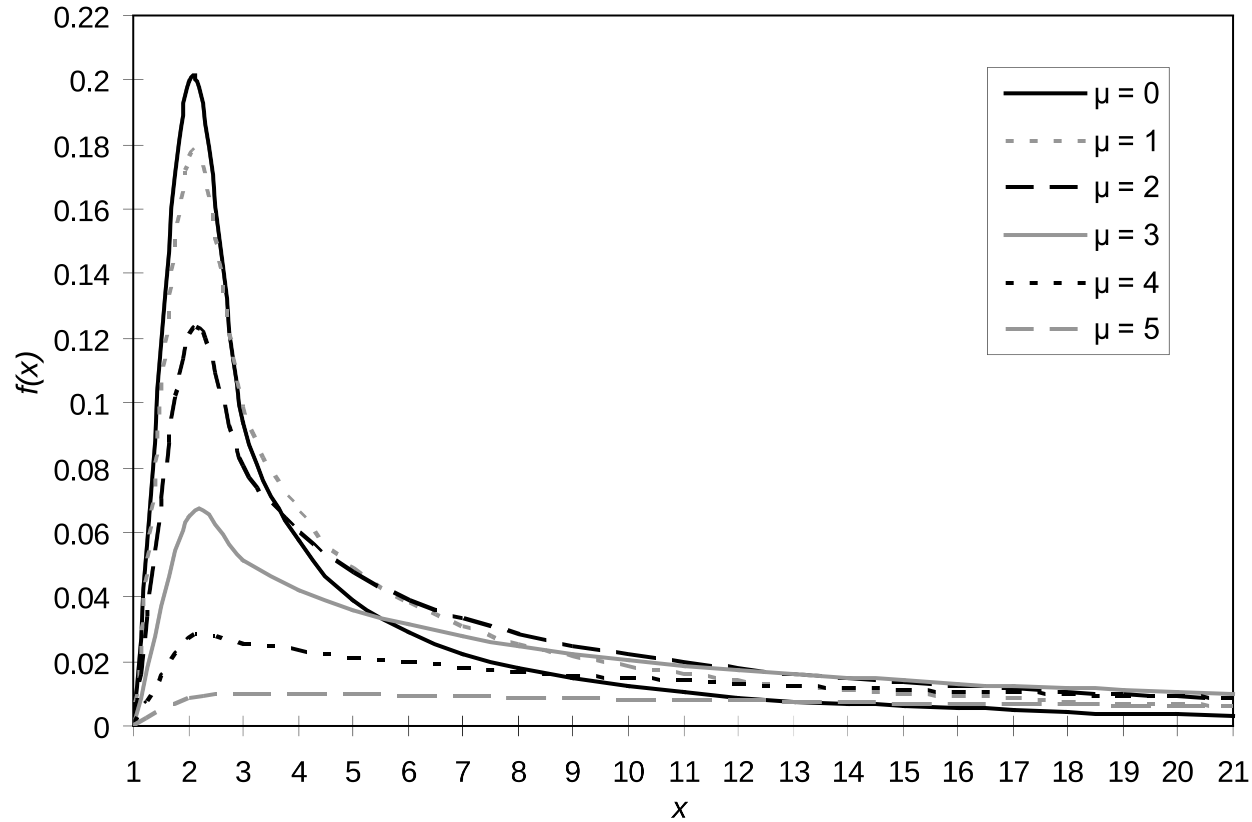

Fig. 2 Probability density function of two-parameter lognormal distribution for the parameter value σ = 2 ($\sigma^2 = 4$)

Source: Own research

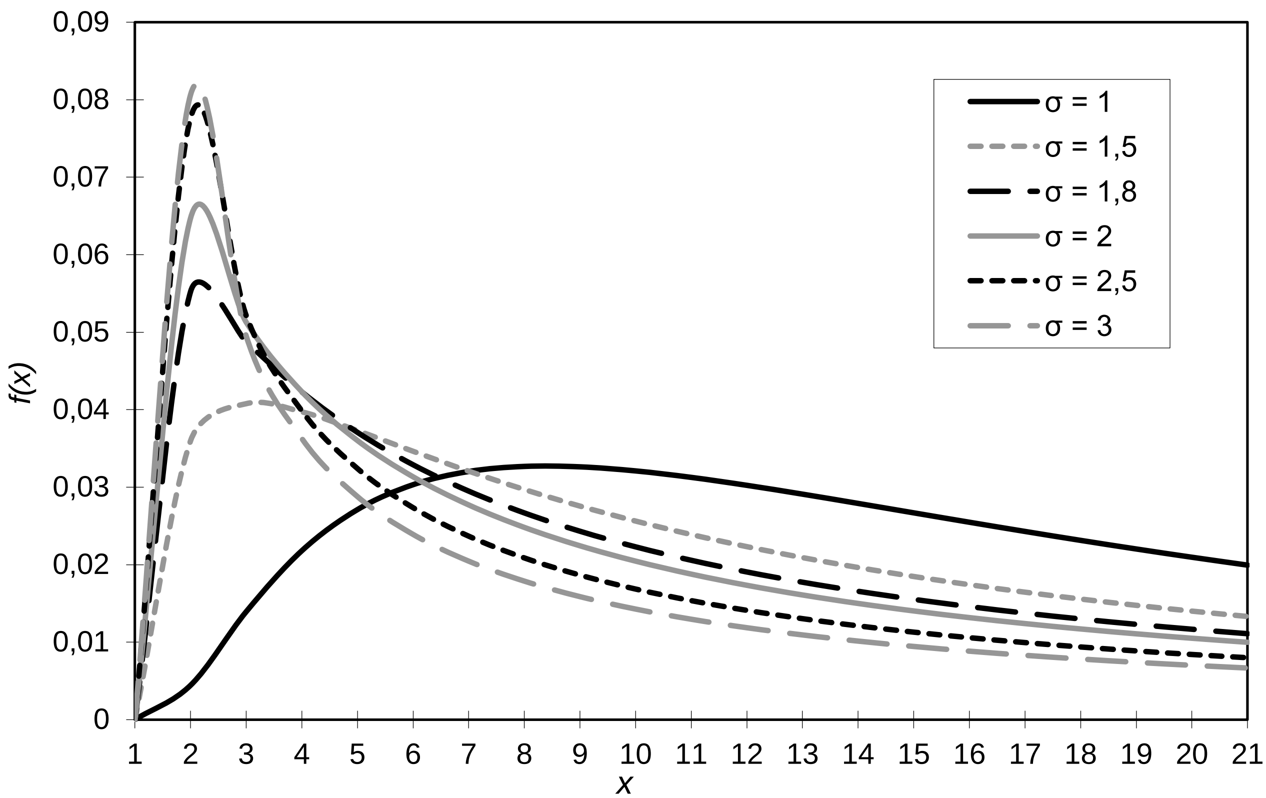

Fig. 3 Probability density function of two-parameter lognormal distribution for the parameter value μ = 3

Source: Own research

Three-Parameter Lognormal Distribution

The random variable X has a three-parameter lognormal distribution with parameters μ, σ^2 and θ, where $-\infty < \mu < \infty$, $\sigma^2 > 0$, $-\infty < \theta < \infty$ if its probability density function has the form

$$f(x;\, \mu, \sigma^2, \theta) = \frac{1}{\sigma\cdot(x-\theta)\cdot\sqrt{2\pi}}\cdot\exp\left[-\frac{[\ln(x-\theta)-\mu]^2}{2\sigma^2}\right], \qquad x > \theta,$$
otherwise equaling 0. (19)

The lognormal distribution with parameters μ, σ^2 and θ (the beginning of the distribution, theoretical minimum) is denoted LN(μ, σ^2, θ). The probability density function of the three-parameter lognormal distribution is again asymmetric, positively skewed. Figs. 4 and 5 show a graph of the probability density function of the three-parameter lognormal distribution LN(μ, σ^2, θ) depending on the values of distribution parameters μ, σ^2 and θ.

The probability density function of the three-parameter lognormal distribution is sometimes given in the form

$$f(x;\, \gamma, \delta, \theta) = \frac{1}{\sigma\cdot(x-\theta)\cdot\sqrt{2\pi}}\cdot\exp\left[-\frac{[\ln(x-\theta)-\mu]^2}{2\sigma^2}\right], \qquad x > \theta,$$
otherwise equaling 0, (20)

and it holds again for the probability density functions (19) and (20) that $\mu = -\frac{\gamma}{\delta}$ and $\sigma = \frac{1}{\delta}$.

Having substituted $\theta = 0$ (distribution minimum) into the formulas for the probability density function of the three-parameter lognormal distribution (19) and (20), we obtain the expressions for the probability density function of the two-parameter lognormal distribution (1) and (2).

The distribution function of the three-parameter lognormal distribution has the form

$$F(x) = \Phi\left[\frac{\ln(x-\theta)-\mu}{\sigma}\right], \quad x > \theta. \qquad (21)$$

If the random variable X has a three-parameter lognormal distribution LN(μ, σ^2, θ), then the random variable

$$Y = \ln(X-\theta) \qquad (22)$$

has a normal distribution N(μ, σ^2) and the random variable

$$U = \frac{\ln(X-\theta)-\mu}{\sigma} = \gamma + \delta\cdot\ln(X-\theta) \qquad (23)$$

has a standardized normal distribution N(0; 1). Thus, the parameter μ is the expected value of the random variable (22), the parameter σ^2 being its variance. The parameter θ is the beginning of the distribution, i.e. the theoretical minimum of the random variable X.

For $\omega = \exp(\sigma^2)$, the r-th common and central moments of the three-parameter lognormal distribution have the form

$$\mu'_r = E(X^r) = \theta + \exp\left(r\cdot\mu + \frac{r^2\sigma^2}{2}\right), \qquad (24)$$

$$\mu_r = E[(X-\mu'_1)^r] = \omega^{r/2}\cdot\left[\sum_{j=0}^{r}(-1)^j\cdot\binom{r}{j}\cdot\omega^{(r-j)\cdot(r-j-1)/2}\right]\cdot\exp(r\cdot\mu), \qquad (25)$$

specifically again,

$$\mu_3 = \omega^{3/2}\cdot(\omega-1)^2\cdot(\omega+2)\cdot\exp(3\cdot\mu), \qquad (26)$$

$$\mu_4 = \omega^2\cdot(\omega-1)^2\cdot(\omega^4+2\omega^3+3\omega^2-3)\cdot\exp(4\cdot\mu). \qquad (27)$$

From (24) and (25), we obtain the expressions for the expected value and variance of the random variable X having a three-parameter lognormal distribution

$$E(X) = \theta + \exp\left(\mu + \frac{\sigma^2}{2}\right), \qquad (28)$$

$$D(X) = \exp(2\mu+\sigma^2)\cdot[\exp(\sigma^2)-1] = \exp(2\mu)\cdot\omega\cdot(\omega-1). \qquad (29)$$

The expression for the median

$$Median(X) = \theta + \exp(\mu) \qquad (30)$$

is based on the formula for a 100 · P% quantile of this distribution

$$x_P = \theta + \exp(\mu + \sigma\cdot u_P). \qquad (31)$$

The three-parameter lognormal distribution is unimodal, with one mode

$$Mode(X) = \theta + \exp(\mu-\sigma^2) = \theta + \frac{\exp(\mu)}{\omega}. \qquad (32)$$

The relation between the expected value, median and mode,

$$E(X) > Median(X) > Mode(X), \qquad (33)$$

which is typical especially for a positively skewed frequency distribution, results again from the equations (28), (30) and (32).

However, the coefficient of variation of the three-parameter lognormal distribution is the function of all three distribution parameters μ, σ^2 and θ, see [9],

$$V(X) = \frac{\exp\left(\mu+\frac{\sigma^2}{2}\right)\sqrt{\exp(\sigma^2)-1}}{\theta+\exp\left(\mu+\frac{\sigma^2}{2}\right)} = \frac{\exp\left(\mu+\frac{\sigma^2}{2}\right)\sqrt{\omega-1}}{\theta+\exp\left(\mu+\frac{\sigma^2}{2}\right)}. \qquad (34)$$

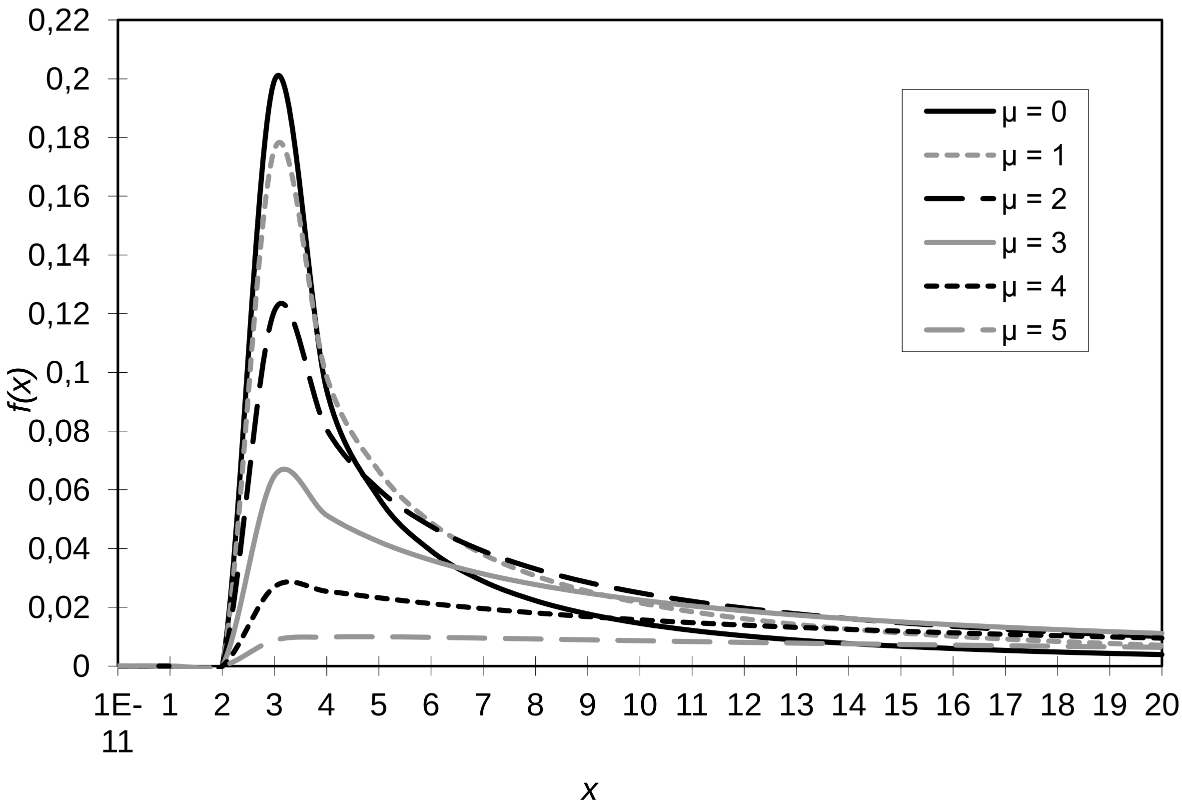

Fig. 4 Probability density function of three-parameter lognormal distribution for parameter values σ = 2 ($\sigma^2 = 4$); θ = 2

Source: Own research

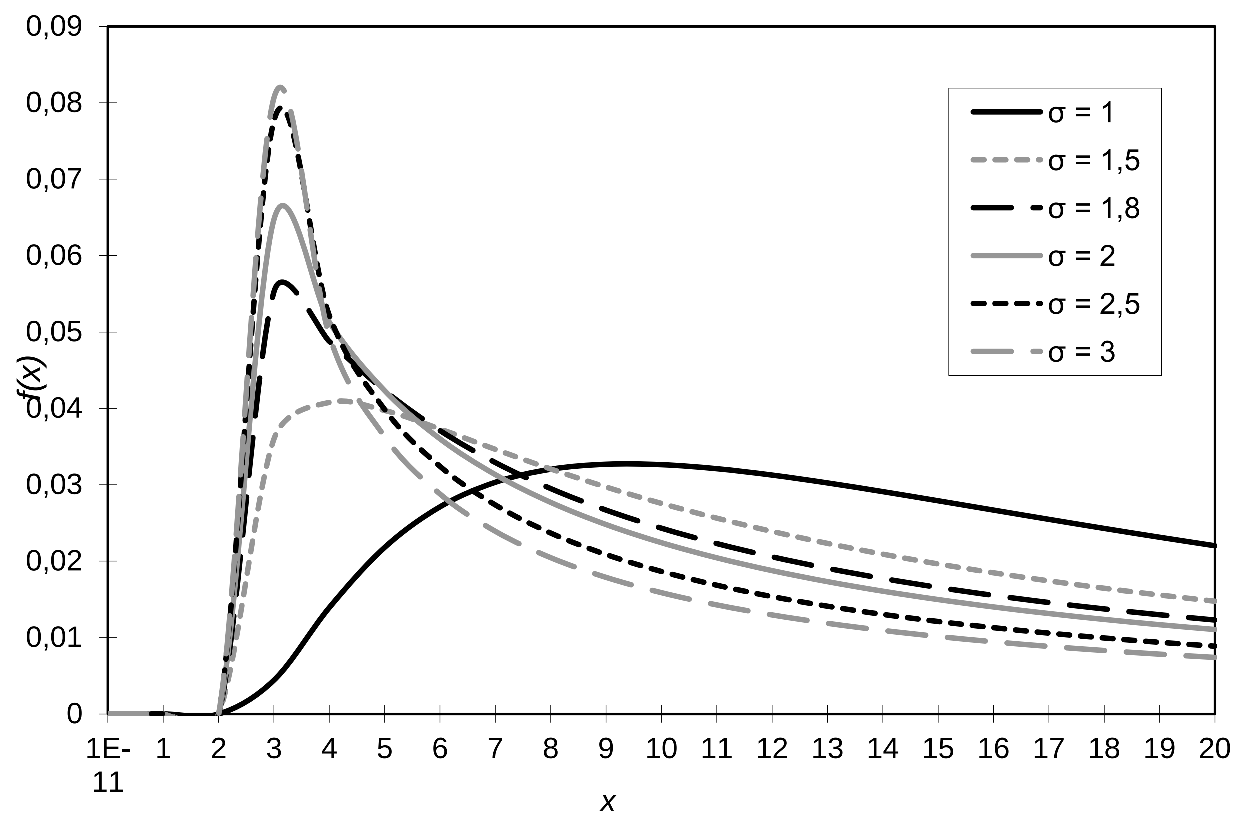

Fig. 5 Probability density function of three-parameter lognormal distribution for parameter values μ = 3; θ = 2

Source: Own research

The Gini coefficient also depends on the values of all three distribution parameters μ, σ^2 and θ, see [9],

$$G = \frac{\exp\left(\mu + \frac{\sigma^2}{2}\right) \cdot \operatorname{erf}\left(\frac{\sigma}{2}\right)}{\theta + \exp\left(\mu + \frac{\sigma^2}{2}\right)}. \quad (35)$$

Moment measures of the skewness and kurtosis depend on a single parameter σ^2

$$\beta_1 = \sqrt{\exp(\sigma^2) - 1} \cdot [\exp(\sigma^2) + 2] = \sqrt{\omega - 1} \cdot (\omega + 2)\,, \quad (36)$$

$$\beta_2 = [\exp(4\sigma^2) + 2\exp(3\sigma^2) + 3\exp(2\sigma^2) - 3] = (\omega^4 + 2\omega^3 + 3\omega^2 - 3). \quad (37)$$

Four-Parameter Lognormal Distribution

The random variable X has a four-parameter lognormal distribution with parameters μ, σ^2, θ and τ, where $-\infty < \mu < \infty$, $\sigma^2 > 0$, $-\infty < \theta < \tau < \infty$ if its probability density function has the form

$$f(x;\, \mu, \sigma^2, \theta, \tau) = \frac{(\tau - \theta)}{\sigma \cdot (x - \theta) \cdot (\tau - x) \cdot \sqrt{2\pi}} \cdot \exp\left[-\frac{\left(\ln \frac{x - \theta}{\tau - x} - \mu\right)^2}{2\sigma^2} \right], \qquad \theta < x < \tau,$$

otherwise equaling 0. (38)

The lognormal distribution with parameters μ, σ^2, θ and τ is denoted LN(μ, σ^2, θ, τ). The probability density function of the four-parameter lognormal distribution LN(μ, σ^2, θ, τ) can have very different shapes depending on the values of distribution parameters; see Figs. 6−8. The distribution can be also bimodal for $\sigma^2 > 2$ and $|\mu| < \sigma^2 \cdot \sqrt{(1 - 2/\sigma^2)} - 2\tanh^{-1}\sqrt{(1 - 2/\sigma^2)}$.

The probability density function of the four-parameter lognormal distribution is sometimes given in the form

$$f(x;\, \gamma, \delta, \theta, \tau) = \frac{\delta \cdot (\tau - \theta)}{(x - \theta) \cdot (\tau - x) \cdot \sqrt{2\pi}} \cdot \exp\left[-\frac{1}{2}\left(\gamma + \delta \cdot \ln \frac{x - \theta}{\tau - x}\right)^2 \right], \qquad \theta < x < \tau,$$

otherwise equaling 0. (39)

and it holds again for the probability density functions (38) and (39) that $\mu = -\frac{\gamma}{\delta}$ and $\sigma = \frac{1}{\delta}$.

If the random variable X has a four-parameter lognormal distribution LN(μ, σ^2, θ, τ), then the random variable

$$Y = \ln \frac{X - \theta}{\tau - X} \quad (40)$$

has a normal distribution N(μ, σ^2), the random variable

$$U = \frac{\ln \frac{X - \theta}{\tau - X} - \mu}{\sigma} = \gamma + \delta \cdot \ln \frac{X - \theta}{\tau - X} \quad (41)$$

having a standardized normal distribution N(0; 1).

The parameter μ is thus the expected value of the random variable (40), parameter σ^2 being its variance. The parameter θ is the beginning of the distribution of the random variable X (theoretical minimum), parameter τ representing its endpoint (theoretical maximum).

B. L-moments and TL-moments

L-moments of Probability Distribution

Let X be a continuous random variable that has a distribution with the distribution function $F(x)$ and quantile function $x(F)$. Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the order statistics of a random sample of the sample size n, coming from the distribution of the random variable X. L-moment of the r-th order of the random variable X is defined as

$$\lambda_r = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot E(X_{r-j:r}), \quad r = 1, 2, \ldots. \quad (42)$$

The expected value of the r-th order statistic of a random sample of size n has the form

$$E(X_{r:n}) = \frac{n!}{(r-1)! \cdot (n-r)!} \cdot \int_0^1 x(F) \cdot [F(x)]^{r-1} \cdot [1 - F(x)]^{n-r} \mathrm{d}F(x). \quad (43)$$

If we substitute equation (43) into equation (42), we obtain after adjustments

$$\lambda_r = \int_0^1 x(F) \cdot P^*_{r-1}[F(x)]\, \mathrm{d}F(x), \quad r = 1, 2, \ldots, \quad (44)$$

where

$$P^*_r[F(x)] = \sum_{j=0}^{r} p^*_{r,j} \cdot [F(x)]^j \quad \text{and} \quad p^*_{r,j} = (-1)^{r-j} \cdot \binom{r}{j} \cdot \binom{r+j}{j}, \quad (45)$$

and $P^*_r[F(x)]$ is the r-th shifted Legendre polynomial. Substituting expression (43) into expression (42), we also obtain

$$\lambda_r = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \frac{r!}{(r-j-1)! \cdot j!} \cdot \int_0^1 x(F) \cdot [F(x)]^{r-j-1} \cdot [1 - F(x)]^j \, \mathrm{d}F(x), \quad r = 1, 2, \ldots. \quad (46)$$

The letter "L" in "L-moments" indicates that the r-th L-moment λ_r is a linear function of the expected value of a certain linear combination of the order statistics. The actual estimation of the r-th L-moment λ_r based on the obtained data sample is then a linear combination of order data values, i.e. L-statistics.

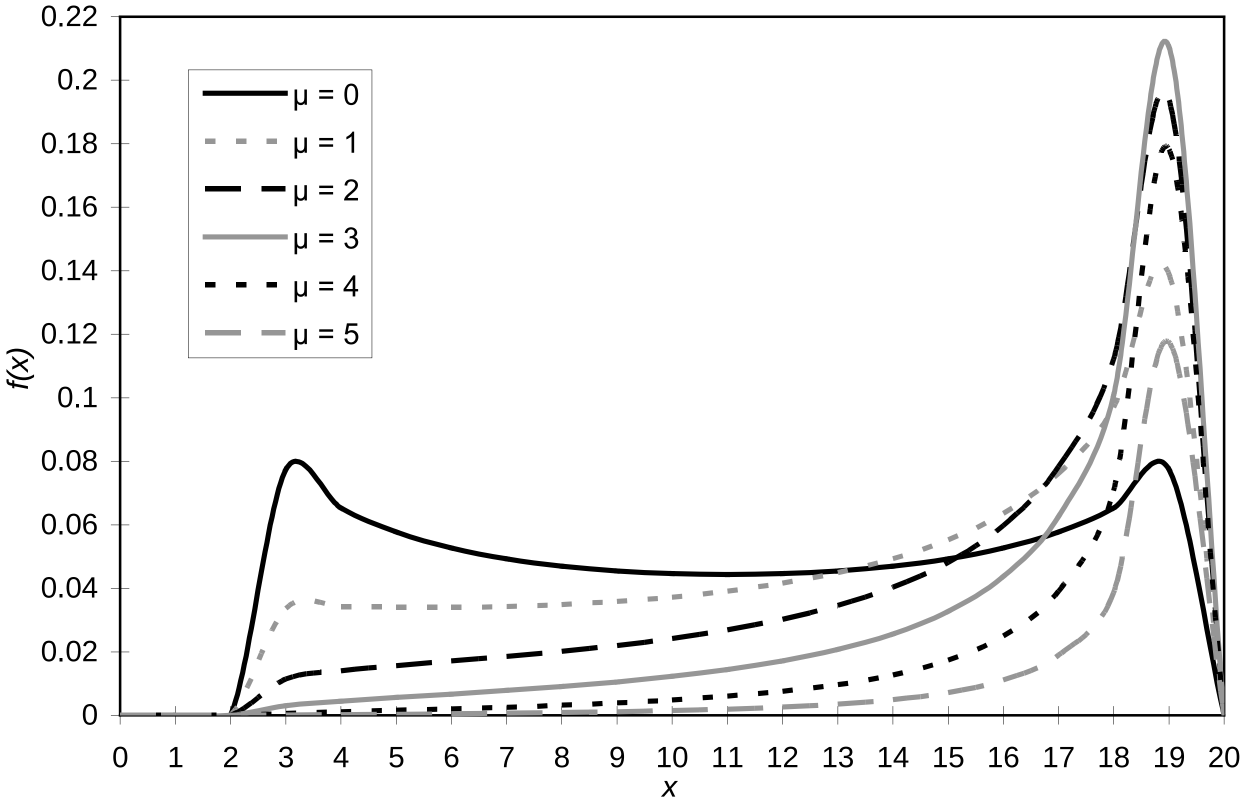

Fig. 6 Probability density function of four-parameter lognormal distribution for parameter values σ = 2 ($\sigma^2 = 4$); θ = 2; τ = 20

Source: Own research

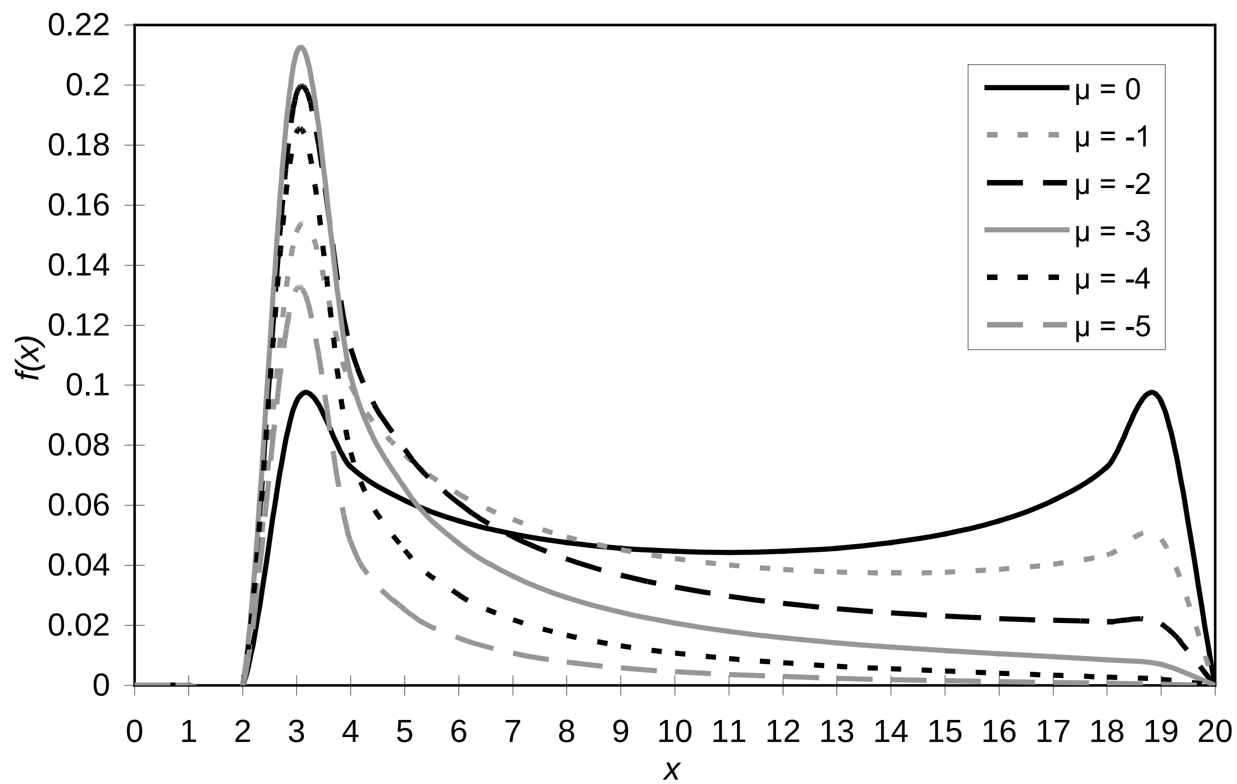

Fig. 7 Probability density function of four-parameter lognormal distribution for parameter values σ = 2 ($\sigma^2 = 4$); θ = 2; τ = 20

Source: Own research

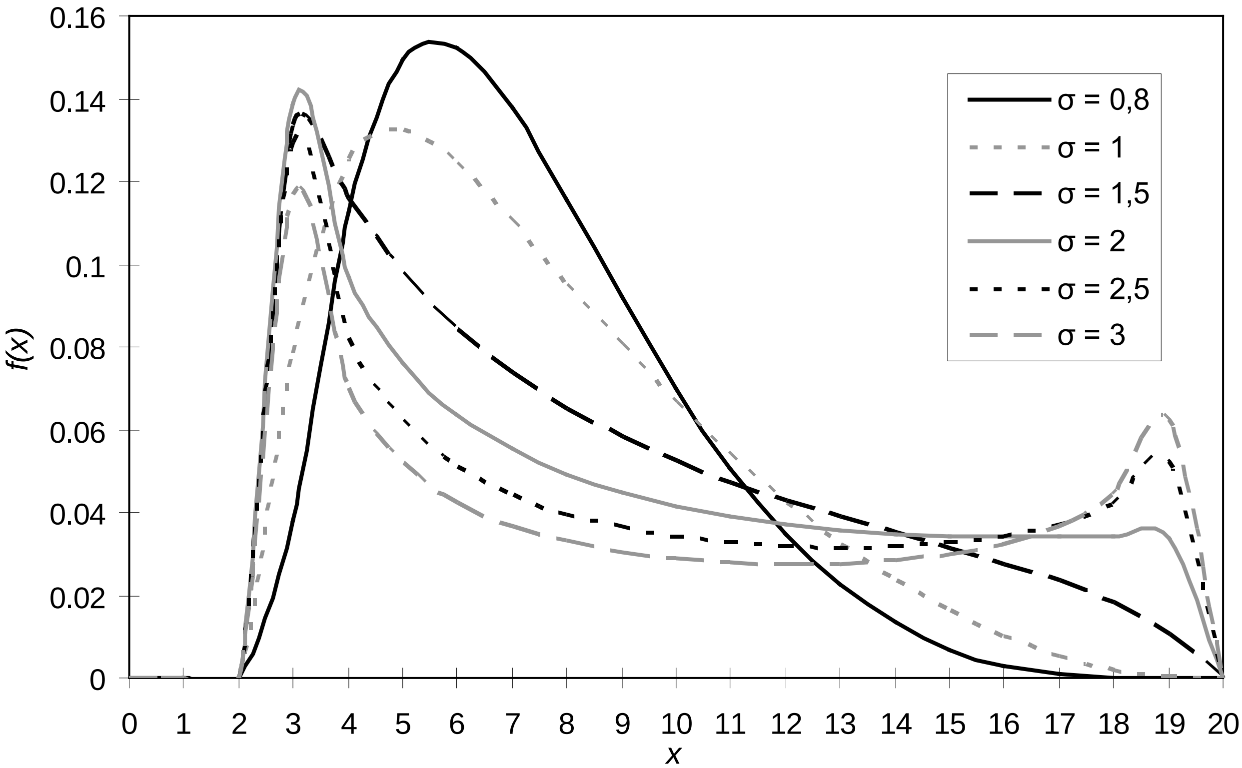

Fig. 8 Probability density function of four-parameter lognormal distribution for parameter values μ = –1; θ = 2; τ = 20

Source: Own research

The first four L-moments of the probability distribution are now defined as

$$\lambda_1 = E(X_{1:1}) = \int_0^1 x(F)\, \mathrm{d}\, F(x), \tag{47}$$

$$\lambda_2 = \frac{1}{2} E(X_{2:2} - X_{1:2}) = \int_0^1 x(F) \cdot [2\, F(x) - 1]\, \mathrm{d}\, F(x), \tag{48}$$

$$\lambda_3 = \frac{1}{3} E(X_{3:3} - 2X_{2:3} + X_{1:3}) = \int_0^1 x(F) \cdot \{6[F(x)]^2 - 6\, F(x) + 1\}\, \mathrm{d}\, F(x), \tag{49}$$

$$\lambda_4 = \frac{1}{4} E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}) = \int_0^1 x(F) \cdot \{20[F(x)]^3 - 30[F(x)]^2 + 12[F(x)] - 1\}\, \mathrm{d}\, F(x). \tag{50}$$

The probability distribution can be specified by its L-moments, even if some of its conventional moments do not exist, the opposite, however, not being true. It can be proved that the first L-moment λ_1 is a characteristic of the location and the second L-moment λ_2 is that of variability. It is often desirable to standardize higher L-moments λ_r, $r \geq 3$, so that they can be independent on specific units of the random variable X. The ratio of L-moments of the r-th order of the random variable X is defined as

$$\tau_r = \frac{\lambda_r}{\lambda_2}, \quad r = 3, 4, \ldots. \tag{51}$$

We can also define a function of L-moments which is analogous to the classical coefficient of variation, i.e. the so called L-coefficient of variation

$$\tau = \frac{\lambda_2}{\lambda_1}. \tag{52}$$

The ratio of L-moments τ_3 is the skewness characteristic, the ratio of L-moments τ_4 being the kurtosis characteristic of the respective probability distribution. The main probability distribution properties are very well summarized by the following four characteristics: L-location λ_1, L-variability λ_2, L-skewness τ_3 and L-kurtosis τ_4. L-moments λ_1 and λ_2, L-coefficient of variation τ and ratios of L-moments τ_3 and τ_4 are the most useful characteristics allowing us to summarize the probability distribution. Their main properties are existence (if the expected value of the distribution exists, then all its L-moments exist) and uniqueness (if the expected value of the distribution exists, then L-moments define the only one distribution, i.e. no two distributions have the same L-moments).

Using the equations (47)−(50) and (51), we obtain the expressions for L-moments and the ratios of L-moments for the chosen probability distributions, respectively; see Table XII.

Table XII Formulas for the distribution function or quantile function, L-moments and ratios of L-moments of chosen probability distributions

Distribution	Distribution function $F(x)$ or quantile function $x(F)$	L-moments and ratios of L-moments
Uniform	$x(F) = \alpha + (\beta - \alpha) \cdot F(x)$	$\lambda_1 = \frac{\alpha + \beta}{2}$ $\lambda_2 = \frac{\beta - \alpha}{6}$ $\tau_3 = 0$ $\tau_4 = 0$
Exponential	$x(F) = \xi - \alpha \cdot \ln[1 - F(x)]$	$\lambda_1 = \xi + \alpha$ $\lambda_2 = \frac{\alpha}{2}$ $\tau_3 = \frac{1}{3}$ $\tau_4 = \frac{1}{6}$
Gumbel	$x(F) = \xi - \alpha \cdot \ln[-\ln F(x)]$	$\lambda_1 = \xi + e \cdot \alpha$ $\lambda_2 = \alpha \cdot \ln 2$ $\tau_3 = 0,1699$ $\tau_4 = 0,1504$
Logistic	$x(F) = \xi + \alpha \cdot \ln \frac{F(x)}{1 - F(x)}$	$\lambda_1 = \xi$ $\lambda_2 = \alpha$ $\tau_3 = 0$ $\tau_4 = \frac{1}{6}$
Normal	$F(x) = \Phi\left[\frac{x(F) - \mu}{\sigma}\right]$	$\lambda_1 = \mu$ $\lambda_2 = \pi^{-1} \cdot \sigma$ $\tau_3 = 0$ $\tau_4 = 30 \cdot \pi^{-1} \cdot (\tan \sqrt{2})^{-1} - 9 = 0,1226$
General Pareto	$x(F) = \xi + \alpha \cdot \frac{1 - [1 - F(x)]^k}{k}$	$\lambda_1 = \xi + \frac{\alpha}{1 + k}$ $\lambda_2 = \frac{\alpha}{(1 + k) \cdot (2 + k)}$ $\tau_3 = \frac{1 - k}{3 + k}$ $\tau_4 = \frac{(1 - k) \cdot (2 - k)}{(3 + k) \cdot (4 + k)}$

Source: [14], own research

Table XII Continuation

Distribution	Distribution function $F(x)$ or quantile function $x(F)$	L-moments and ratios of L-moments
General extermal	$x(F)=\xi+\alpha\cdot\frac{1-[-\ln F(x)]^k}{k}$	$\lambda_1=\xi+\alpha\cdot\frac{1-\Gamma(1+k)}{k}$ $\lambda_2=\alpha\cdot\frac{(1-2^{-k})\cdot\Gamma(1+k)}{k}$ $\tau_3=\frac{2\cdot(1-3^{-k})}{1-2^{-k}}-3$ $\tau_4=\frac{1-6{,}2^{-k}+10{,}3^{-k}-5{,}4^{-k}}{1-2^{-k}}$
General logistic	$x(F)=\xi+\alpha\cdot\frac{1-\left[\frac{1-F(x)}{F(x)}\right]^k}{k}$	$\lambda_1=\xi+\alpha\cdot\frac{1-\Gamma(1+k)\cdot\Gamma(1-k)}{k}$ $\lambda_2=\alpha\cdot\Gamma(1+k)\cdot\Gamma(1-k)$ $\tau_3=-k$ $\tau_4=\frac{1+5k^2}{6}$
Lognormal	$F(x)=\Phi\left\{\frac{\ln[x(F)-\xi]-\mu}{\sigma}\right\}$	$\lambda_1=\xi+\exp\left(\mu+\frac{\sigma^2}{2}\right)$ $\lambda_2=\exp\left(\mu+\frac{\sigma^2}{2}\right)\cdot\operatorname{erf}\left(\frac{\sigma}{2}\right)$ $\tau_3=6\pi^{-\frac{1}{2}}\cdot\frac{\int_0^{\frac{\sigma}{2}}\operatorname{erf}\left(\frac{x}{\sqrt{3}}\right)\cdot\exp(-x^2)\,\mathrm{d}x}{\operatorname{erf}\left(\frac{\sigma}{2}\right)}$
Gamma	$F(x)=\frac{\beta^{-\alpha}}{\Gamma(\alpha)}\cdot\int_0^{x(F)}t^{\alpha-1}\cdot\exp\left(-\frac{t}{\beta}\right)\mathrm{d}t$	$\lambda_1=\alpha\cdot\beta$ $\lambda_2=\pi^{-\frac{1}{2}}\cdot\beta\cdot\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)}$ $\tau_3=6I_{\frac{1}{3}}(\alpha,2\alpha)-3$ [1]

Source: [14], own research

Sample L-moments

L-moments are usually estimated from a random sample drawn from the unknown distribution. Since the r-th L-moment λ_r is a function of order statistics expected values of the r-sized random sample, it is naturally estimated using the so-called U-statistic, i.e. the corresponding function of the sample order statistics (averaged over all subsets of the sample size r that may be formed from the obtained random sample of size n).

Let $x_1, x_2, \ldots, x_n$ be a sample and $x_{1:n}\le x_{2:n}\le\ldots\le x_{n:n}$ an order sample. Then the r-th sample L-moment can be written as

$$l_r=\binom{n}{r}^{-1}\sum_{1\le i_1<i_2<\ldots<i_r\le n}\ldots\sum\frac{1}{r}\cdot\sum_{j=0}^{r-1}(-1)^j\cdot\binom{r-1}{j}\cdot x_{i_{r-j}:n},\quad r=1,2,\ldots,n. \qquad (53)$$

Hence the first four sample L-moments have the form

$$l_1=\frac{1}{n}\cdot\sum_i x_i, \qquad (54)$$

$$l_2=\frac{1}{2}\cdot\binom{n}{2}^{-1}\cdot\sum\sum_{i>j}(x_{i:n}-x_{j:n}), \qquad (55)$$

[1] $I_x(p,q)$ **is an incomplete beta function**

$$l_3 = \frac{1}{3} \cdot \binom{n}{3}^{-1} \cdot \sum_{i>j>k}\sum\sum (x_{i:n} - 2x_{j:n} + x_{k:n}), \tag{56}$$

$$l_4 = \frac{1}{4} \cdot \binom{n}{4}^{-1} \cdot \sum_{i>j>k>l}\sum\sum\sum (x_{i:n} - 3x_{j:n} + 3x_{k:n} - x_{l:n}). \tag{57}$$

U-statistics are widely used, especially in nonparametric statistics. Their positive properties are the absence of bias, asymptotic normality and a slight resistance due to the influence of outliers.

When calculating the r-th sample L-moment, it is not necessary to repeat the process over all subsets of the sample size r; this statistic can be expressed directly as a linear combination of the order statistics of a random sample of size n.

The estimation of $E(X_{r:r})$ obtained using U-statistics can be written as $r \cdot b_{r-1}$, where

$$b_r = \frac{1}{n} \cdot \binom{n-1}{r}^{-1} \cdot \sum_{j=r+1}^{n} \binom{j-1}{r} \cdot x_{j:n}, \tag{58}$$

namely

$$b_0 = \frac{1}{n} \cdot \sum_{j=1}^{n} x_{j:n}, \tag{59}$$

$$b_1 = \frac{1}{n} \cdot \sum_{j=2}^{n} \frac{(j-1)}{(n-1)} \cdot x_{j:n}, \tag{60}$$

$$b_2 = \frac{1}{n} \cdot \sum_{j=3}^{n} \frac{(j-1)\cdot(j-2)}{(n-1)\cdot(n-2)} \cdot x_{j:n}, \tag{61}$$

therefore generally,

$$b_r = \frac{1}{n} \cdot \sum_{j=r+1}^{n} \frac{(j-1)\cdot(j-2)\cdot\ldots\cdot(j-r)}{(n-1)\cdot(n-2)\cdot\ldots\cdot(n-r)} \cdot x_{j:n}. \tag{62}$$

The first sample L-moments can be denoted as

$$l_1 = b_0, \tag{63}$$

$$l_2 = 2b_1 - b_0, \tag{64}$$

$$l_3 = 6b_2 - 6b_1 + b_0, \tag{65}$$

$$l_4 = 20b_3 - 30b_2 + 12b_1 - b_0. \tag{66}$$

Overall, we can therefore write

$$l_{r+1} = \sum_{k=0}^{r} p^{*}_{r,k} \cdot b_k, \quad r = 0, 1, \ldots, n-1, \tag{67}$$

where

$$p^{*}_{r,k} = (-1)^{r-k} \cdot \binom{r}{k} \cdot \binom{r+k}{k} = \frac{(-1)^{r-k} \cdot (r+k)!}{(k!)^2 \cdot (r-k)!}. \tag{68}$$

The use of sample L-moments is similar to that of sample conventional L-moments. Sample L-moments summarize the basic properties of the sample distribution, namely the location (level), variability, skewness and kurtosis, thus allowing for the estimation of the corresponding properties of the probability distribution from which the sample comes. They can be employed in estimating the parameters of the relevant probability distribution. Sample L-moments are often preferred over conventional moments within such applications since – as the linear functions of sample values – the former are less sensitive to sample variability and measurement errors (in the case of extreme observations) than the latter. L-moments therefore lead to more accurate and robust estimations of parameters (characteristics) of the basic probability distribution.

Sample L-moments have already been used in statistics, however, not as part of a unified theory. The first sample L-moment l_1 is a sample L-location (sample average), the second sample L-moment l_2 being a sample L-variability. Natural estimation of the ratio of L-moments (51) is the sample ratio of L-moments

$$t_r = \frac{l_r}{l_2}, \quad r = 3, 4, \ldots. \tag{69}$$

Hence t_3 is a sample L-skewness and t_4 is a sample L-kurtosis. Sample ratios of L-moments t_3 and t_4 may be used as characteristics of skewness and kurtosis of the sample data set.

The Gini middle difference relates to sample L-moments, having the form

$$G = \binom{n}{2}^{-1} \cdot \sum_{i>j}\sum (x_{i:n} - x_{j:n}), \tag{70}$$

and the Gini coefficient, which depends only on a single parameter σ in the case of a two-parameter lognormal distribution, depending, however, on the values of all three parameters in the case of a three-parameter lognormal distribution. Table XIII presents the expressions for parameter estimations of chosen probability distributions obtained using the method of L-moments. For more details, see, e.g. [1]–[11], [14]–[15], [18] and [26].

Table XIII Formulas for parameter estimation provided by the method of L-moments of chosen probability distributions

Distribution	Parameter estimation
Exponential	(ξ known) $\hat{\alpha} = l_1$
Gumbel	$\hat{\alpha} = \frac{l_2}{\ln 2}$ $\hat{\xi} = l_1 - e \cdot \hat{\alpha}$
Logistic	$\hat{\alpha} = l_2$ $\hat{\xi} = l_1$
Normal	$\hat{\sigma} = \pi^{\frac{1}{2}} \cdot l_2$ $\hat{\mu} = l_1$
General Pareto	(ξ known) $\hat{k} = \frac{l_1}{l_2} - 2$ $\hat{\alpha} = (1+\hat{k}) \cdot l_1$
General extermal	$z = \frac{2}{3+t_3} - \frac{\ln 2}{\ln 3}$ $\hat{k} = 7{,}8590\, z + 2{,}9554\, z^2$ $\hat{\alpha} = \frac{l_2 \cdot \hat{k}}{(1-2^{-\hat{k}}) \cdot \Gamma(1+\hat{k})}$ $\hat{\xi} = l_1 + \hat{\alpha} \cdot \frac{\Gamma(1+\hat{k}) - 1}{\hat{k}}$
General logistic	$\hat{k} = -t_3$ $\hat{\alpha} = \frac{l_2}{\Gamma(1+\hat{k}) \cdot \Gamma(1-\hat{k})}$ $\hat{\xi} = l_1 + \frac{l_2 - \hat{\alpha}}{\hat{k}}$
Gamma	(ξ known) $t = \frac{l_2}{l_1}$ if $0 < t < \frac{1}{2}$, then: $z = \pi \cdot t^2$, $\hat{\alpha} \approx \frac{1 - 0{,}3080\, z}{z - 0{,}058\,12\, z^2 + 0{,}017\,65\, z^3}$ if $\frac{1}{2} \le t < 1$, then: $z = 1 - t$, $\hat{\alpha} \approx \frac{0{,}7213\, z - 0{,}5947\, z^2}{1 - 2{,}1817\, z + 1{,}2113\, z^2}$ $\hat{\beta} = \frac{l_1}{\hat{\alpha}}$

Source: [14], own research

TL-moments of Probability Distribution

An alternative robust version of L-moments will be introduced now. This modification of L-moments is called the “trimmed L-moments” and is noted TL-moments. In this modification of L-moments, the expected values of order statistics of a random sample (in L-moments definition of probability distributions) are replaced by the expected values of order statistics of a larger random sample, the sample size growing in such a way that it corresponds to the total size of the adjustment, as shown below.

TL-moments have certain advantages over conventional L-moments and central moments. TL-moment of probability distribution may exist even if the corresponding L-moment or central moment of this probability distribution does not exist, as it is the case of the Cauchy distribution. Sample TL-moments are more resistant to outliers in the data. The method of TL-moments is not intended to replace the existing robust methods, but rather as their supplement, particularly in situations with outliers in the data.

In this alternative robust modification of L-moments, the expected value $E(X_{r-j:r})$ is replaced by that of $E(X_{r+t_1-j\,:\,r+t_1+t_2})$. For each r, we increase the size of a random sample from the original r to $r + t_1 + t_2$, working only with the expected values of these r modified order statistics $X_{t_1+1:r+t_1+t_2}, X_{t_1+2:r+t_1+t_2}, \ldots, X_{t_1+r:r+t_1+t_2}$ by trimming t_1 and t_2 (the lowest and highest value, respectively, from a conceptual sample). This modification is called the r-th trimmed L-moment (TL-moment) and marked $\lambda_r^{(t_1,t_2)}$. Thus, TL-moment of the r-th order of a random variable X is defined as

$$\lambda_r^{(t_1,t_2)} = \frac{1}{r}\cdot\sum_{j=0}^{r-1}(-1)^j\cdot\binom{r-1}{j}\cdot E(X_{r+t_1-j:r+t_1+t_2}),\quad r = 1, 2, \ldots. \tag{71}$$

It is evident from the expressions (71) and (42) that TL-moments are reduced to L-moments when $t_1 = t_2 = 0$. Although we can also consider applications where the adjustment values are not equal, i.e. $t_1 \neq t_2$, we focus only on the symmetry of $t_1 = t_2 = t$. Then the expression (71) can be rewritten

$$\lambda_r^{(t)} = \frac{1}{r}\cdot\sum_{j=0}^{r-1}(-1)^j\cdot\binom{r-1}{j}\cdot E(X_{r+t-j:r+2t}),\quad r = 1, 2, \ldots. \tag{72}$$

Thus, for example, $\lambda_1^{(t)} = E(X_{1+t:1+2t})$ is the expected value of the median of a conceptual random sample of the sample size $1 + 2t$. It is to be noted here that $\lambda_1^{(t)}$ is equal to zero for distributions symmetric around zero.

For $t = 1$, the first four TL-moments have the form¨

$$\lambda_1^{(1)} = E(X_{2:3}), \tag{73}$$

$$\lambda_2^{(1)} = \frac{1}{2}E(X_{3:4} - X_{2:4}), \tag{74}$$

$$\lambda_3^{(1)} = \frac{1}{3}E(X_{4:5} - 2X_{3:5} + X_{2:5}), \tag{75}$$

$$\lambda_4^{(1)} = \frac{1}{4}E(X_{5:6} - 3X_{4:6} + 3X_{3:6} - X_{2:6}). \tag{76}$$

Measurements of location, variability, skewness and kurtosis of a probability distribution analogous to conventional L-moments (47)−(50) are based on $\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}$ a $\lambda_4^{(1)}$.

The expected value $E(X_{r:n})$ can be written using the formula (43). Applying the equation (43), we can re-express the right side of the equation (72)

$$\lambda_r^{(t)} = \frac{1}{r}\cdot\sum_{j=0}^{r-1}(-1)^j\cdot\binom{r-1}{j}\cdot\frac{(r+2t)!}{(r+t-j-1)!\cdot(t+j)!}\cdot\int_0^1 x(F)\cdot[F(x)]^{r+t-j-1}\cdot[1-F(x)]^{t+j}\,\mathrm{d}F(x),\ r = 1, 2, \ldots. \tag{77}$$

It is necessary to bear in mind that $\lambda_r^{(0)} = \lambda_r$ normally represents the r-th L-moment with no adjustment.

The expressions (73)−(76) for the first four TL-moments ($t = 1$) may be written in an alternative manner

$$\lambda_1^{(1)} = 6\cdot\int_0^1 x(F)\cdot[F(x)]\cdot[1-F(x)]\,\mathrm{d}F(x), \tag{78}$$

$$\lambda_2^{(1)} = 6\cdot\int_0^1 x(F)\cdot[F(x)]\cdot[1-F(x)]\cdot[2F(x)-1]\,\mathrm{d}F(x), \tag{79}$$

$$\lambda_3^{(1)} = \frac{20}{3}\cdot\int_0^1 x(F)\cdot[F(x)]\cdot[1-F(x)]\cdot\{5[F(x)]^2-5F(x)+1\}\,\mathrm{d}F(x), \tag{80}$$

$$\lambda_4^{(1)} = \frac{15}{2}\cdot\int_0^1 x(F)\cdot[F(x)]\cdot[1-F(x)]\cdot\{14[F(x)]^3-21[F(x)]^2+9[F(x)]-1]\,\mathrm{d}F(x). \tag{81}$$

The distribution can be identified by its TL-moments, although some of its L-moments and conventional moments do not exit. For example, $\lambda_1^{(1)}$ (the expected value of the median of a conceptual random sample of size three) exists for the Cauchy distribution, despite the first L-moment λ_1 not existing.

TL-skewness $\tau_3^{(t)}$ and TL-kurtosis $\tau_4^{(t)}$ can be defined analogously as L-skewness τ_3 and L-kurtosis τ_4

$$\tau_3^{(t)} = \frac{\lambda_3^{(t)}}{\lambda_2^{(t)}}, \tag{82}$$

$$\tau_4^{(t)} = \frac{\lambda_4^{(t)}}{\lambda_2^{(t)}}. \tag{83}$$

Sample TL-moments

Let $x_1, x_2, \ldots, x_n$ be the sample and $x_{1:n} \le x_{2:n} \le \ldots \le x_{n:n}$ an order sample. The expression

$$\hat{E}(X_{j+1:j+l+1}) = \frac{1}{\binom{n}{j+l+1}} \cdot \sum_{i=1}^{n} \binom{i-1}{j} \cdot \binom{n-i}{l} \cdot x_{i:n} \quad (84)$$

is considered to be an unbiased estimation of the expected value of the $(j + 1)$-th order statistic $X_{j+1:j+l+1}$ in the conceptual random sample of the sample size $(j + l + 1)$. Now we assume that in the definition of the TL-moment $\lambda_r^{(t)}$ in (72) the expression $E(X_{r+t-j:r+2t})$ is replaced by its unbiased estimation

$$\hat{E}(X_{r+t-j:r+2t}) = \frac{1}{\binom{n}{r+2t}} \cdot \sum_{i=1}^{n} \binom{i-1}{r+t-j-1} \cdot \binom{n-i}{t+j} \cdot x_{i:n}, \quad (85)$$

which is obtained by substituting $j \to r+t-j-1$ and $l \to t+j$ in (84). Now we get the r-th sample TL-moment

$$l_r^{(t)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \hat{E}(X_{r+t-j:r+2t}), \quad r = 1, 2, \ldots, n-2t, \quad (86)$$

i.e.

$$l_r^{(t)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \frac{1}{\binom{n}{r+2t}} \cdot \sum_{i=1}^{n} \binom{i-1}{r+t-j-1} \cdot \binom{n-i}{t+j} \cdot x_{i:n}, \quad r = 1, 2, \ldots, n-2t, \quad (87)$$

which is an unbiased estimation of the r-th TL-moment $\lambda_r^{(t)}$. Let us note that for each $j = 0, 1, \ldots, r-1$, the values $x_{i:n}$ in (87) are not equal to zero only for $r+t-j \le i \le n-t-j$ in relation to combination numbers. Simple adjustment of the equation (87) provides an alternative linear form

$$l_r^{(t)} = \frac{1}{r} \cdot \sum_{i=r+t}^{n-t} \left[\frac{\sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \binom{i-1}{r+t-j-1} \cdot \binom{n-i}{t+j}}{\binom{n}{r+2t}} \right] \cdot x_{i:n}. \quad (88)$$

For example, for the first sample TL-moment (for $r = 1$) we obtain

$$l_1^{(t)} = \sum_{i=t+1}^{n-t} w_{i:n}^{(t)} \cdot x_{i:n}, \quad (89)$$

where the weights are given by

$$w_{i:n}^{(t)} = \frac{\binom{i-1}{t} \cdot \binom{n-i}{t}}{\binom{n}{2t+1}}. \quad (90)$$

The above results can be used to estimate TL-skewness $\tau_3^{(t)}$ and TL-kurtosis $\tau_4^{(t)}$ by simple ratios

$$t_3^{(t)} = \frac{l_3^{(t)}}{l_2^{(t)}}, \quad (91)$$

$$t_4^{(t)} = \frac{l_4^{(t)}}{l_2^{(t)}}. \quad (92)$$

We can choose $t = n\alpha$ which represents the amount of adjustment from each end of the sample, α being a certain ratio, where $0 \le \alpha < 0{,}5$.

Table XIV contains the expressions and ratios for TL-moments and expressions for parameter estimations of chosen probability distributions obtained using the method of TL-moments ($t = 1$); for more, see, e.g. [12].

C. *Maximum Likelihood Method*

Let the random sample of the sample size n come from a three-parameter lognormal distribution with the probability density function (19). The likelihood function then has the form

$$L(\boldsymbol{x}; \mu, \sigma^2, \theta) = \prod_{i=1}^{n} f(x_i; \mu, \sigma^2, \theta) =$$

$$= \frac{1}{(\sigma^2)^{n/2} \cdot (2\pi)^{n/2} \cdot \prod_{i=1}^{n} (x_i - \theta)} \cdot \exp\left\{ \sum_{i=1}^{n} -\frac{[\ln(x_i - \theta) - \mu]^2}{2\sigma^2} \right\}. \quad (93)$$

We determine the natural logarithm of the likelihood function

$$\ln L(\boldsymbol{x}; \mu, \sigma^2, \theta) = \sum_{i=1}^{n} -\frac{[\ln(x_i - \theta) - \mu]^2}{2\sigma^2} - \frac{n}{2} \cdot \ln\sigma^2 - \frac{n}{2} \cdot \ln(2\pi) - \sum_{i=1}^{n} \ln(x_i - \theta). \quad (94)$$

We put the first partial derivations of the logarithm of the likelihood function according to μ and σ^2 in equality to zero, obtaining a system of likelihood equations

$$\frac{\partial \ln L(\boldsymbol{x}; \mu, \sigma^2, \theta)}{\partial \mu} = \frac{\sum_{i=1}^{n} [\ln(x_i - \theta) - \mu]}{\sigma^2} = 0, \quad (95)$$

$$\frac{\partial \ln L(\boldsymbol{x}; \mu, \sigma^2, \theta)}{\partial \sigma^2} = \frac{\sum_{i=1}^{n} [\ln(x_i - \theta) - \mu]^2}{2\sigma^4} - \frac{n}{2\sigma^2} = 0. \quad (96)$$

After adjustment, we obtain maximum likelihood estimations of the parameters μ and σ^2 for the parameter θ

Table XIV Formulas for TL-moments and ratios of TL-moments and formulas for parameter estimations made by the method of TL-moments of chosen probability distributions (t = 1)

Distribution	TL-moments and ratios of TL-moments	Parameter estimation
Normal	$\lambda_1^{(1)} = \mu$ $\lambda_2^{(1)} = 0{,}297\,\sigma$ $\tau_3^{(1)} = 0$ $\tau_4^{(1)} = 0{,}062$	$\hat{\mu} = l_1^{(1)}$ $\hat{\sigma} = \frac{l_2^{(1)}}{0{,}297}$
Logistic	$\lambda_1^{(1)} = \mu$ $\lambda_2^{(1)} = 0{,}500\,\sigma$ $\tau_3^{(1)} = 0$ $\tau_4^{(1)} = 0{,}083$	$\hat{\mu} = l_1^{(1)}$ $\hat{\sigma} = 2\,l_2^{(1)}$
Cauchy	$\lambda_1^{(1)} = \mu$ $\lambda_2^{(1)} = 0{,}698\,\sigma$ $\tau_3^{(1)} = 0$ $\tau_4^{(1)} = 0{,}343$	$\hat{\mu} = l_1^{(1)}$ $\hat{\sigma} = \frac{l_2^{(1)}}{0{,}698}$
Exponential	$\lambda_1^{(1)} = \frac{5\alpha}{6}$ $\lambda_2^{(1)} = \frac{\alpha}{4}$ $\tau_3^{(1)} = \frac{2}{9}$ $\tau_4^{(1)} = \frac{1}{12}$	$\hat{\alpha} = \frac{6\,l_1^{(1)}}{5}$

Source: [12], own research

$$\hat{\mu}(\theta) = \frac{\sum_{i=1}^{n} \ln(x_i - \theta)}{n}, \tag{97}$$

$$\hat{\sigma}^2(\theta) = \frac{\sum_{i=1}^{n} [\ln(x_i - \theta) - \hat{\mu}(\theta)]^2}{n}. \tag{98}$$

If the value of the parameter θ is known, we get maximum likelihood estimations of the remaining two parameters of three-parametric lognormal distribution using the equations (97) and (98). However, if the value of the parameter θ is unknown, the problem is more complicated. It can be proved that if the parameter θ closes to $\min\{X_1, X_2, \ldots, X_n\}$, then the maximum likelihood approaches infinity. The maximum likelihood method is often combined with Cohen method, where we put the smallest sample value to be equal to a $100 \cdot (n+1)^{-1}$-percentage quantile

$$x_{\min}^{V} = \hat{\theta} + \exp(\hat{\mu} + \hat{\sigma} \cdot u_{(n+1)^{-1}}). \tag{99}$$

The equation (99) is then combined with the system of equations (97) and (98).

For solving maximum likelihood equations (97) and (98) it is also possible to use $\hat{\theta}$ satisfying the equation

$$\sum_{i=1}^{n}(x_i - \hat{\theta}) + \frac{\sum_{i=1}^{n} \frac{z_i^{\backslash}}{(x_i - \hat{\theta})}}{\hat{\sigma}(\hat{\theta})} = 0, \tag{100}$$

where

$$z_i^{\backslash} = \frac{\ln(x_i - \hat{\theta}) - \hat{\mu}(\hat{\theta})}{\hat{\sigma}(\hat{\theta})}, \tag{101}$$

where $\hat{\mu}(\hat{\theta})$ and $\hat{\sigma}(\hat{\theta})$ satisfy the equations (97) and (98) with the parameter θ replaced by $\hat{\theta}$. We may also obtain the limits of variances

$$n \cdot D(\hat{\theta}) = \frac{\sigma^2 \cdot \exp(2\mu)}{\omega \cdot \left[\omega \cdot (1 + \sigma^2) - 2\sigma^2 - 1\right]}, \tag{102}$$

$$n \cdot D(\hat{\mu}) = \frac{\sigma^2 \cdot [\omega \cdot (1 + \sigma^2) - 2\sigma^2]}{\omega \cdot (1 + \sigma^2) - 2\sigma^2 - 1}, \tag{103}$$

$$n \cdot D(\hat{\sigma}) = \frac{\sigma^2 \cdot [\omega \cdot (1 + \sigma^2) - 1]}{\omega \cdot (1 + \sigma^2) - 2\sigma^2 - 1}. \tag{104}$$

D. Estimation Accuracy Evaluation

It is also necessary to assess the suitability of the constructed model or choose a model from several alternatives. The choice is made using a criterion – either the sum of absolute deviations of the observed and theoretical frequencies for all intervals

$$S = \sum_{i=1}^{k} |n_i - n\pi_i| \tag{105}$$

or the criterion χ^2

$$\chi^2 = \sum_{i=1}^{k} \frac{(n_i - n\pi_i)^2}{n\pi_i}, \tag{106}$$

which is known, where n_i are the observed frequencies in individual income or wage intervals, π_i are the theoretical probabilities of the statistical unit belonging to the i-th interval, $n \cdot \pi_i$ are the theoretical frequencies in individual income or wage intervals, $i = 1, 2, ..., k$, n is a total sample size of the corresponding statistical set and k is the number of intervals.

However, the question of the appropriateness of the model curve for income or wage distribution is not a common mathematical and statistical issue in which we test the null hypothesis

(H_0: the sample coming from an assumed theoretical distribution)

against an alternative hypothesis

(H_1: non H_0),

because in the case of income or wage distribution we often work with large sample sizes and the goodness of fit tests would therefore almost always lead to the rejection of the null hypothesis. This follows not only from the fact that for such large samples, the power of the test is so high at the chosen level of significance that it reveals even the slightest deviations of the model and a given income or wage distribution, but it also results from the way the test is constructed.

Practically, however, we are not interested in such small deviations. Thus, only rough agreement of the model with reality is sufficient and we, so to say, "borrow" the model (curve), the test criterion χ^2 being applied only tentatively. When evaluating the suitability of the model, we proceed subjectively to a large extent, relying on experience and logical analysis.

Estimation of the value of the parameter θ (beginning of the distribution, theoretical minimum) is negative in some cases. It means that a three-parameter lognormal curve shifts to negative values in terms of income or wage at the beginning of its course. Since the curve is at first very close to the horizontal axis, it does not necessarily deny reasonable agreement between the model and the real distribution.

IV. Results and Discussion

A. Income Distribution

The method of TL-moments provided the most accurate results in almost all cases, with negligible exceptions. The method of L-moments proved to be the second most accurate in more than half of the cases. The differences between the L-moments method and that of maximum likelihood, however, are not significant enough to reflect in the number of cases when the former method yielded better results than the latter one. Table XV shows distinctive results for all 168 income distributions, encompassing the total set of households in the Czech Republic. It contains the estimated values of the parameters of the three-parameter lognormal distribution obtained simultaneously using TL-moments, L-moments and maximum likelihood methods as well as the value of the test criterion (106). It is obvious from the criterion values that the method of L-moments brought more accurate results than that of maximum likelihood in four out of seven cases. The most accurate results were produced applying the method of TL-moments in all the seven cases.

Figs. 9–11 allow us to compare these methods in terms of model probability density functions in the chosen years (1992, 2004 and 2007) for the total set of households in the Czech Republic. It is to be noted that in order to enhance the readability of information, in Fig. 9 there is a different scale on the vertical axis than in Figs. 10 and 11; soon after the transformation of the Czech economy from a centrally planned to market system, the income distribution was still exhibiting different characteristics (lower level and variability, higher skewness and kurtosis) from those displayed in recent years. It is clear from the three figures that the methods of TL-moments and L-moments yield very similar results, while the probability density function with the parameters estimated by the maximum likelihood method differs a lot from probability density function models constructed using the first two methods.

Fig. 12 also provides a comparison of the accuracy of these three methods of point parameter estimation. It represents the development of the sample median and theoretical medians of lognormal distribution with the parameters estimated using the methods of TL-moments, L-moments and maximum likelihood again for the total set of households of the Czech Republic in the research period. It is also obvious from this figure that the curve indicating the course of theoretical medians of lognormal distribution with the parameters estimated by TL-moments and L-moments methods are closer to that showing the course of the sample median compared with the curve representing the development of the theoretical median of lognormal distribution with the parameters estimated by the maximum likelihood method.

Table XV Parameter estimations of three-parameter lognormal curves obtained using three various robust methods of point parameter estimation and the value of χ^2 criterion

	Method of TL-moments			Method of L-moments			Maximum likelihood method		
Year	μ	σ^2	θ	μ	σ^2	θ	μ	σ^2	θ
1992	9.722	0.521	14,881	9.696	0.700	14,491	10.384	0.390	-325
1996	10.334	0.573	25,981	10.343	0.545	25,362	10.995	0.424	52.231
2002	10.818	0.675	40,183	10.819	0.773	37,685	11.438	0.459	73.545
2004	10.961	0.552	39,899	11.028	0.675	33,738	11.503	0.665	7.675
2005	11.006	0.521	40,956	11.040	0.677	36,606	11.542	0.446	-8.826
2006	11.074	0.508	44,941	11.112	0.440	40,327	11.623	0.435	-42.331
2007	11.156	0.472	48,529	11.163	0.654	45,634	11.703	0.421	-171.292
Year	Criterion χ^2			Criterion χ^2			Criterion χ^2		
1992	739.512			811.007			1,227.325		
1996	1,503.878			1,742.631			2,197.251		
2002	998.325			1,535.557			1,060.891		
2004	494.441			866.279			524.478		
2005	731.225			899.245			995.855		
2006	831.667			959.902			1,067.789		
2007	1,050.105			1,220.478			1,199.035		

Source: Own research

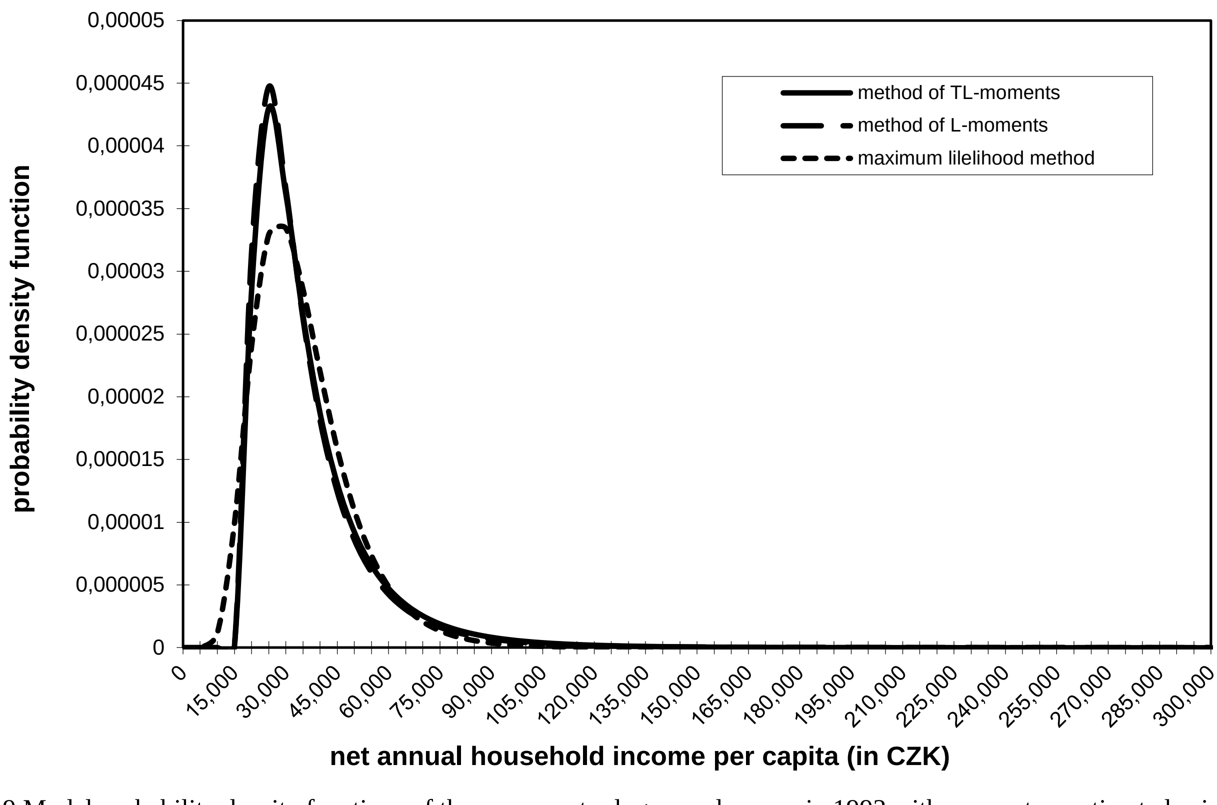

Fig. 9 Model probability density functions of three-parameter lognormal curves in 1992 with parameters estimated using three various robust methods of point parameter estimation

Source: Own research

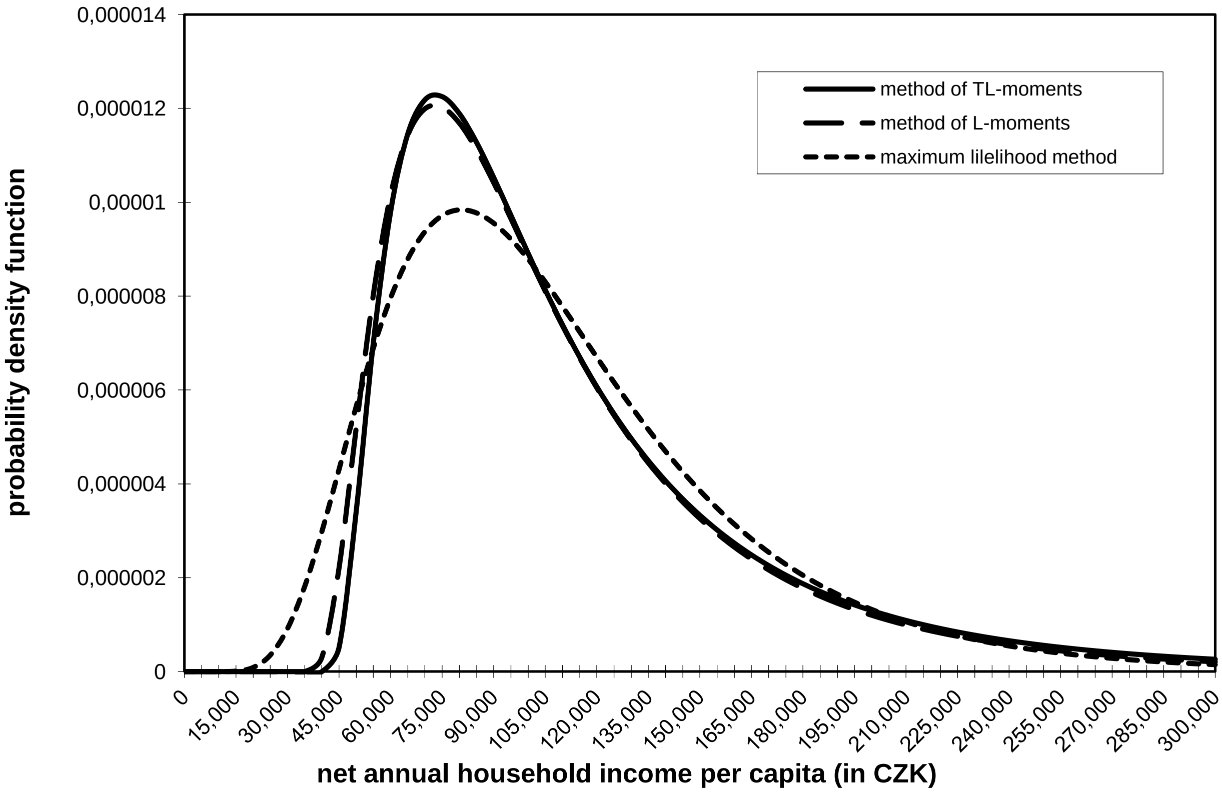

Fig. 10 Model probability density functions of three-parameter lognormal curves in 2004 with parameters estimated using three various robust methods of point parameter estimation

Source: Own research

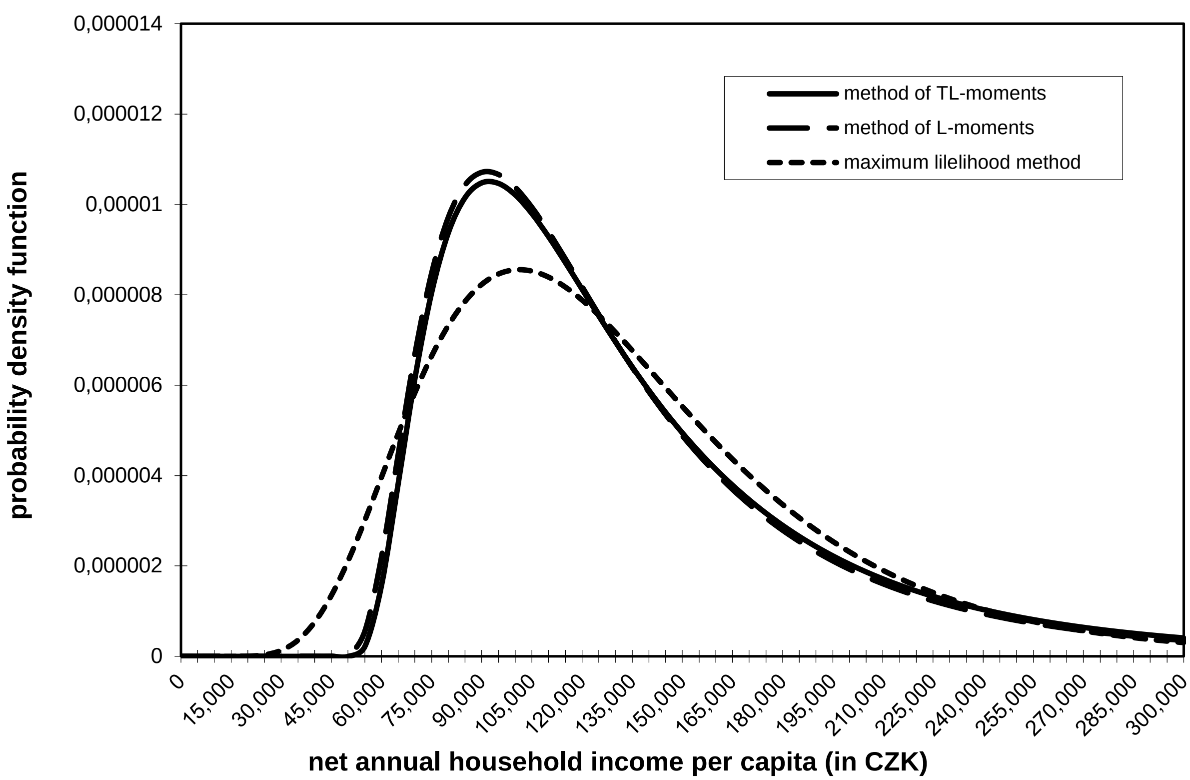

Fig. 11 Model probability density functions of three-parameter lognormal curves in 2007 with parameters estimated using three various robust methods of point parameter estimation

Source: Own research

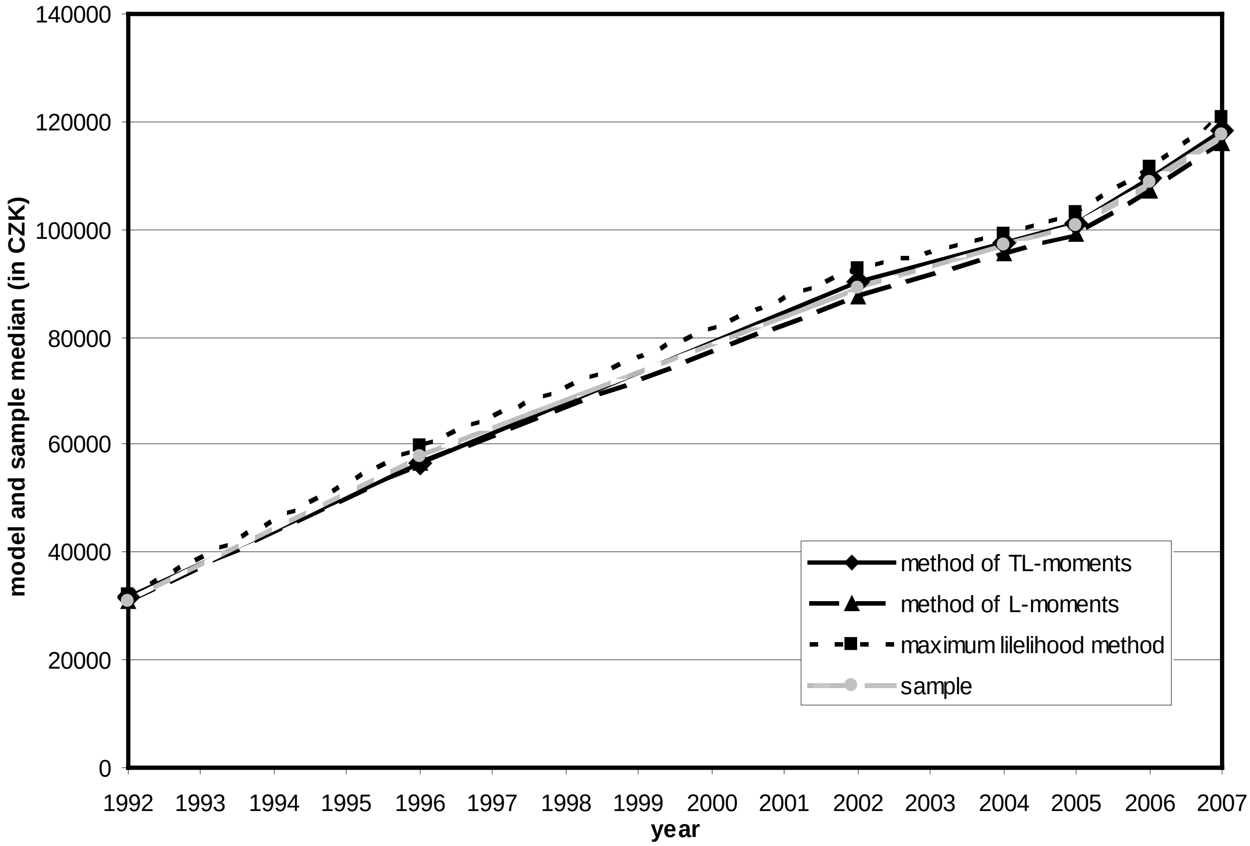

Fig. 12 Development of the model and sample median of net annual household income per capita (in CZK)

Source: Own research

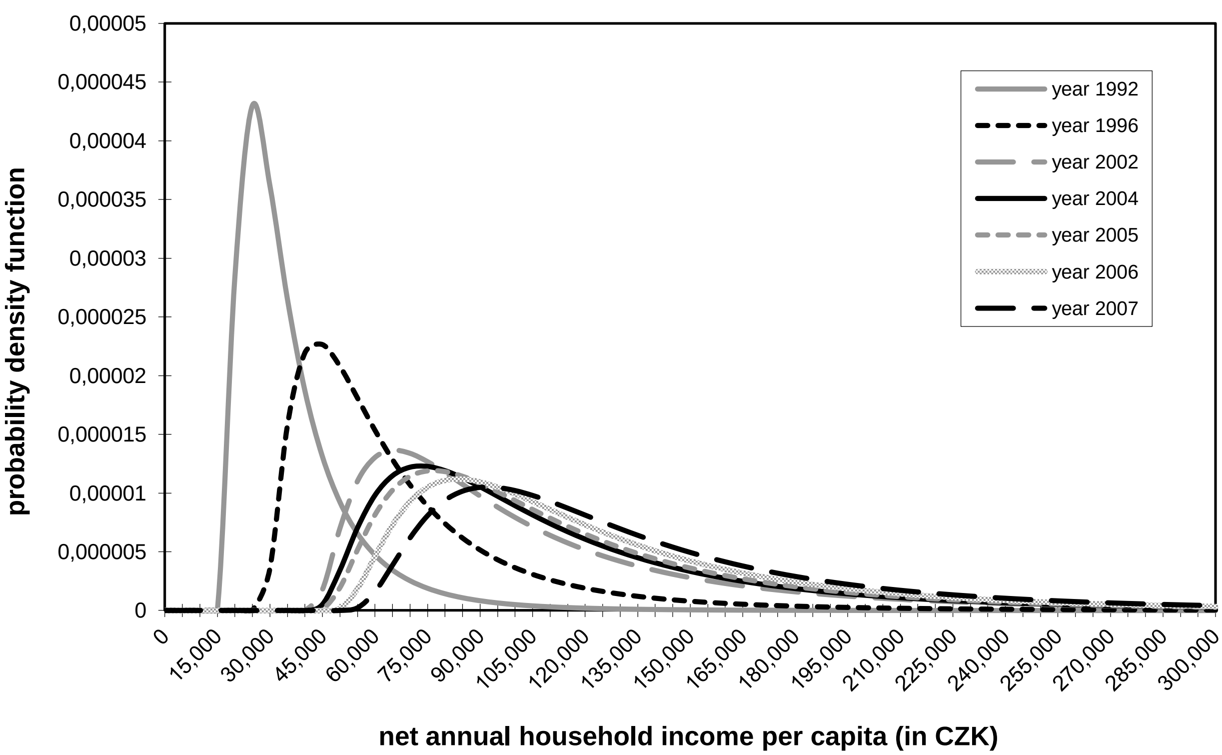

Fig. 13 Development of the probability density function of three-parameter lognormal curves with parameters estimated using the method of TL-moments

Source: Own research

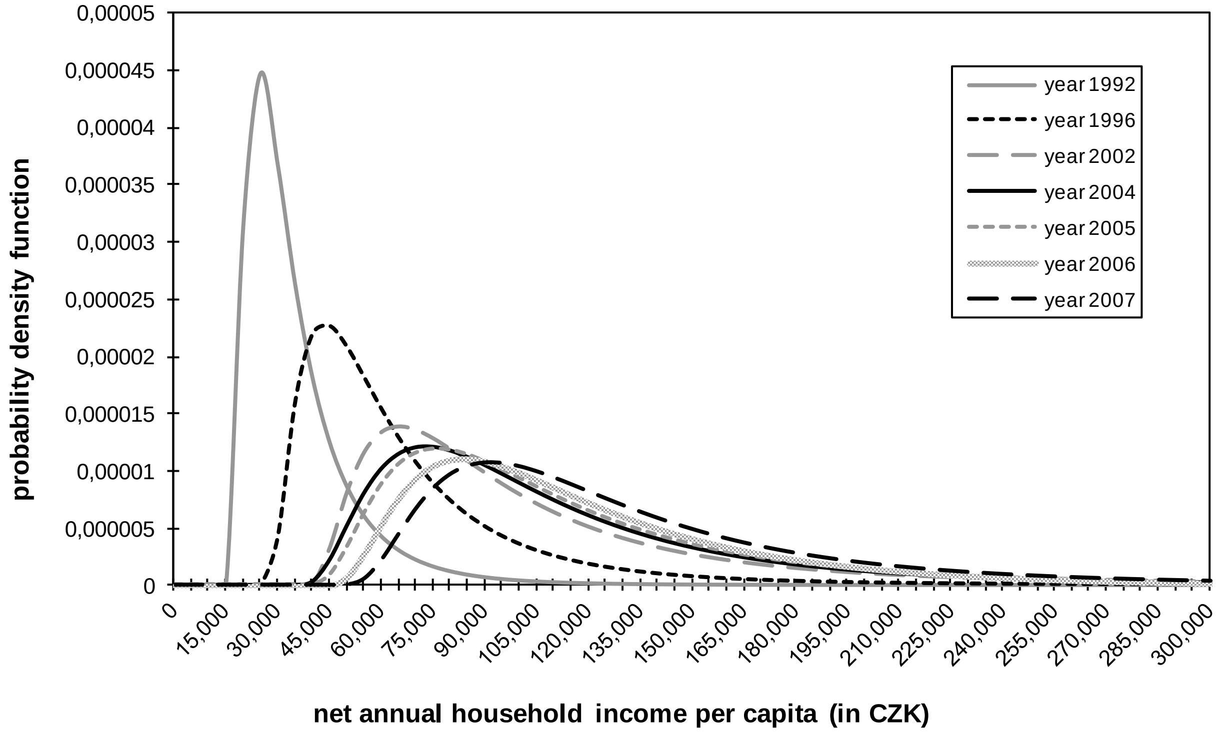

Fig. 14 Development of the probability density function of three-parameter lognormal curves with parameters estimated using the method of L-moments

Source: Own research

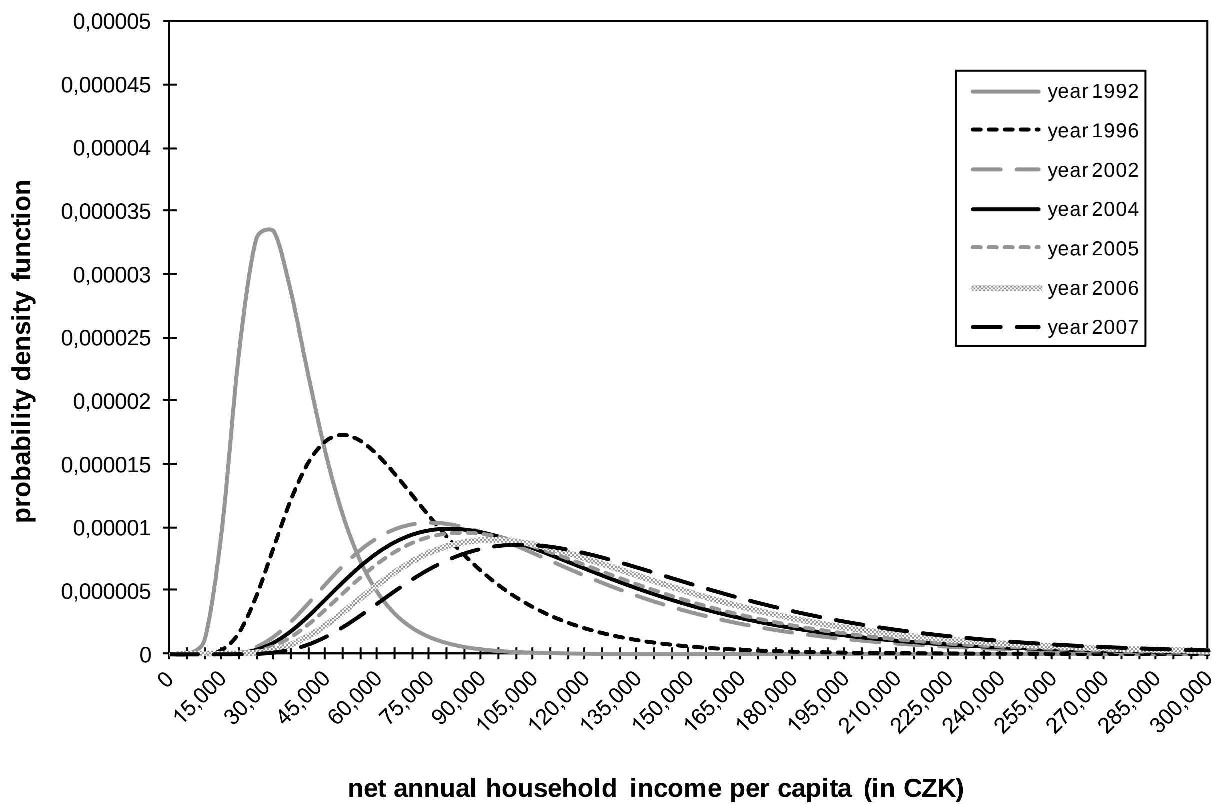

Fig. 15 Development of the probability density function of three-parameter lognormal curves with parameters estimated using the maximum likelihood method

Source: Own research

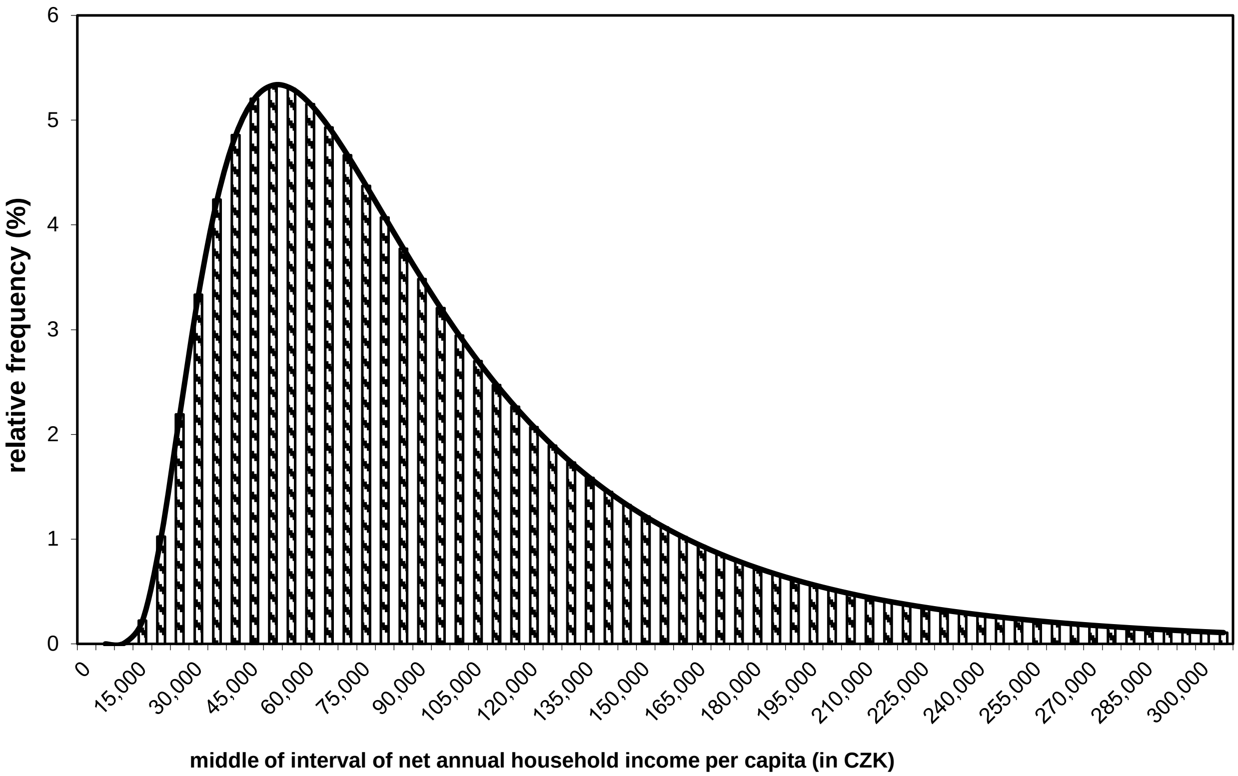

Fig. 16 Model employee ratios (in %) by the bracket of net annual household income per capita with parameters of three-parameter lognormal curves estimated by the method of TL-moments in 2007

Source: Own research

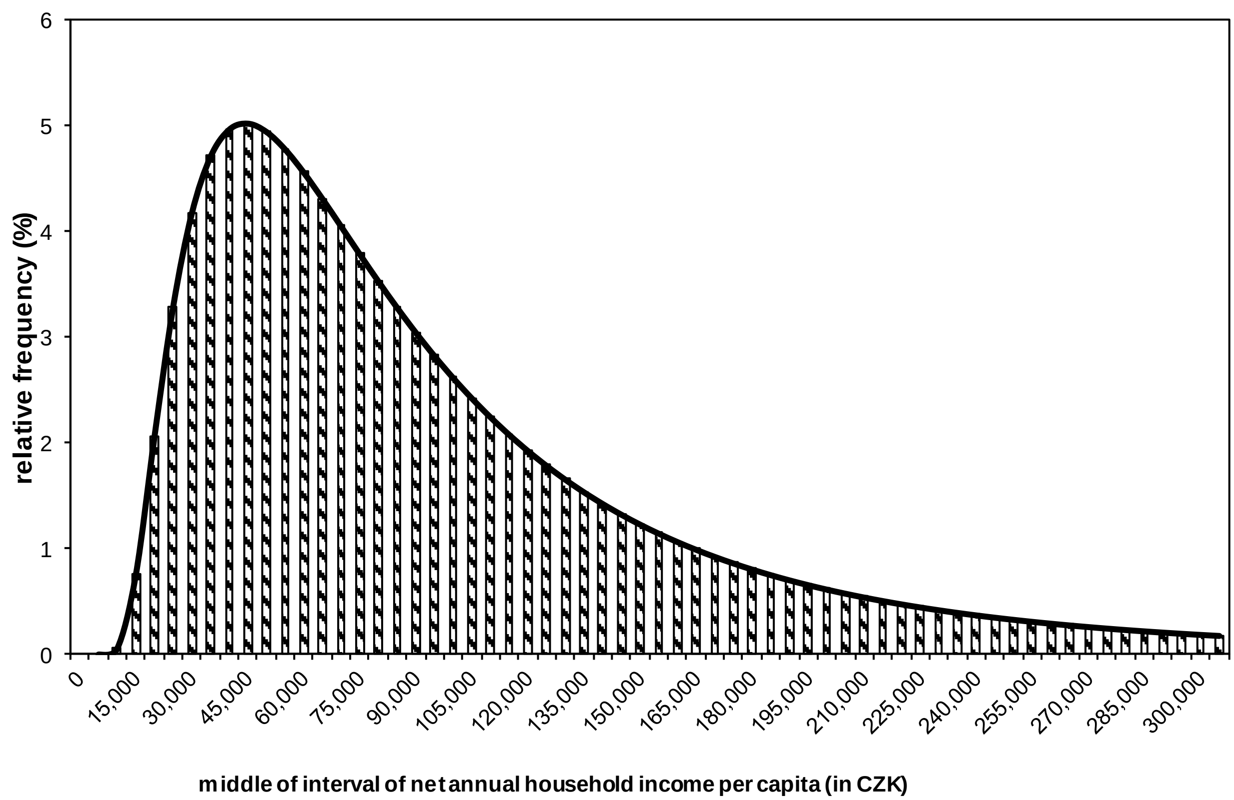

Fig. 17 Model employee ratios (in %) by the bracket of net annual household income per capita with parameters of three-parameter lognormal curves estimated by the method of L-moments in 2007

Source: Own research

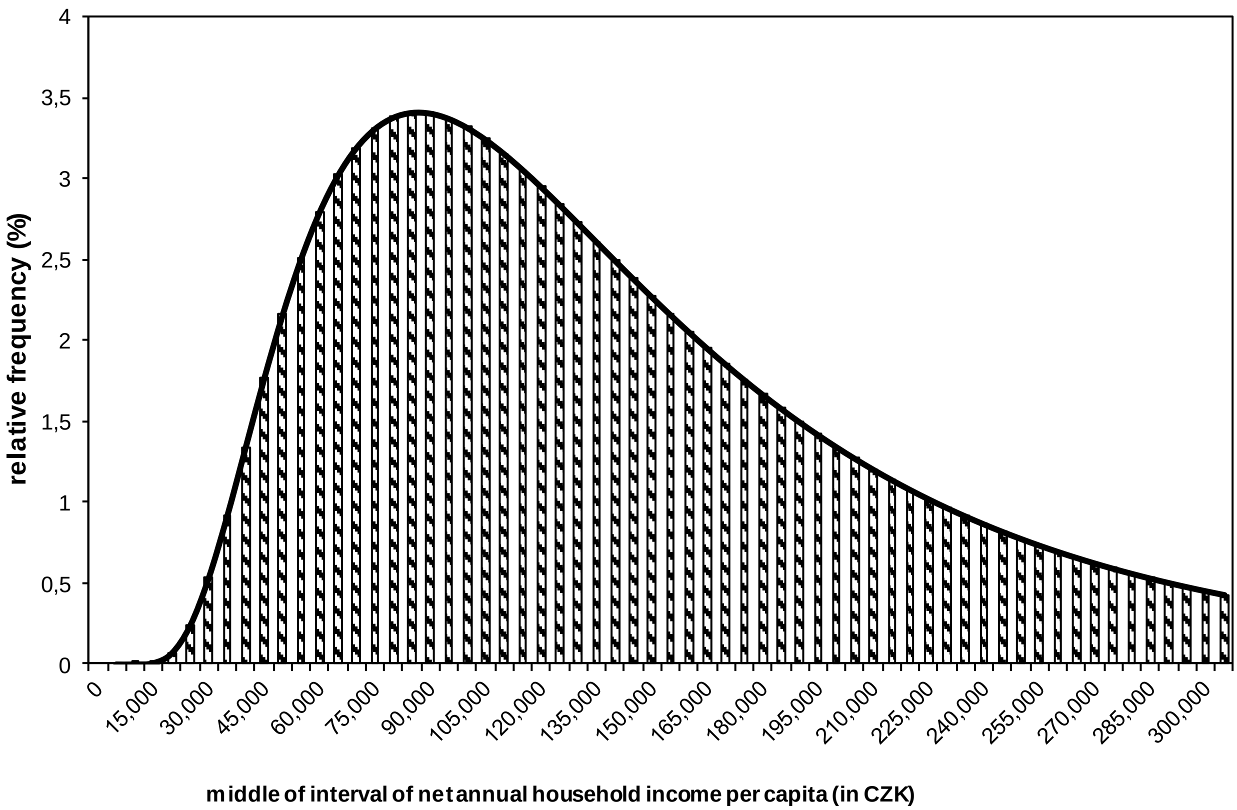

Fig. 18 Model employee ratios (in %) by the bracket of net annual household income per capita with parameters of three-parameter lognormal curves estimated by the maximum likelihood method in 20**07**

Source: Own research

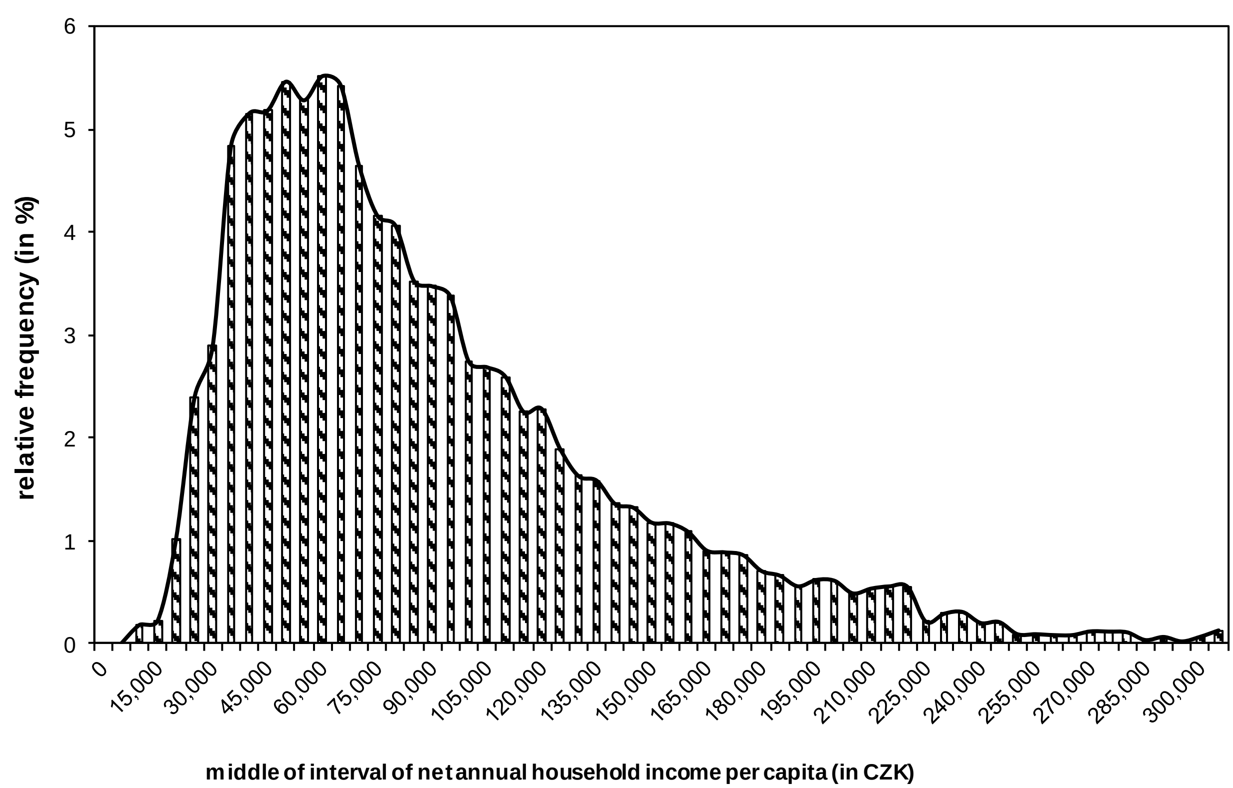

Fig. 19 Sample employee ratios (in %) by the bracket of net annual household income per capita in 2007

Source: Own research

Figs. 13–15 present the development of probability density function models of the three-parameter lognormal distribution again with the parameters estimated using the three methods of parameter estimation for the total set of households in the Czech Republic. In view of these figures, the income distribution in 1992 differs greatly from income distributions in the years to come. We can also observe a certain similarity of the results produced applying the methods of TL-moments and L-moments as well as a considerable divergence between the results obtained using these two methods and those achieved by the maximum likelihood method.

Figs. 16–18 present model relative frequencies (in %) of employees divided according to net annual household per capita income brackets in 2007 obtained using three-parameter lognormal curves with the parameters estimated by TL-moments, L-moments and maximum likelihood methods. These figures also allow us to compare the accuracy of the analyzed methods of point parameter estimation, Fig. 19 showing the actually observed relative frequencies in particular income intervals obtained from a sample.

B. Wage Distribution

Figs. 20 and 21 provide an overview of the development of the annual growth rate of the level of gross monthly wage in the Czech Republic in the research period and the outline of the development of the average annual inflation. Because the growth rate is calculated from the growth coefficient, which is the ratio of two consecutive values of the time series, we would have needed 2002 data to calculate the growth rate for the year 2003. Since 2002 is not included in the analyzed period, the growth rate for 2003 is not presented here. An impact of the global economic downturn on the development of the wage level and inflation in the Czech Republic is evident from these figures. It is apparent from Fig. 20 that having dropped to almost zero in 2009, the annual growth rate of middle gross monthly wage increased a little over the following year, remaining still far below pre-crisis values. It is clear from Fig. 21 that having declined significantly in 2009, the average annual inflation rate rose slightly again during 2010.

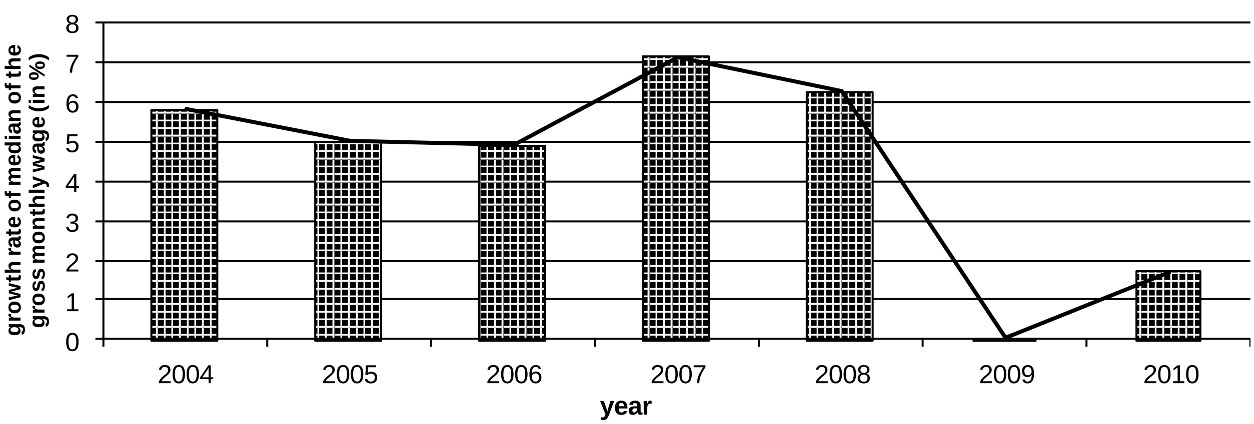

Fig. 20 Annual growth rate of the median of gross monthly wage in the Czech Republic in 2003−2010 (%)

Source: Own research

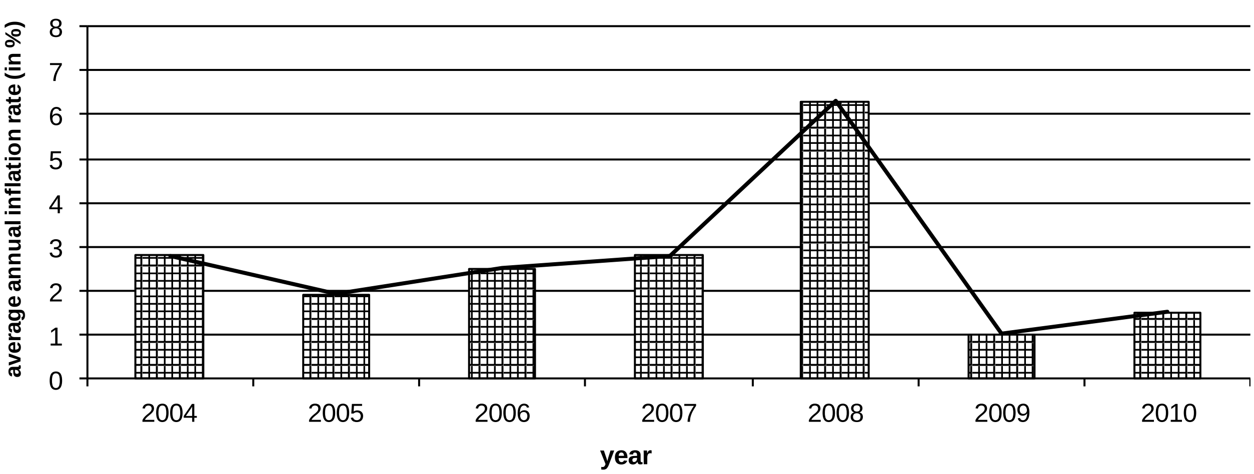

Fig. 21 Average annual inflation rate in 2003−2010 (%)

Source: Own research

Table XVI Parameter estimations obtained using the three methods of point parameter estimation and the value of *S* criterion for total wage distribution in the Czech Republic

	Method of TL-moments			Method of L-moments			Maximum likelihood method		
Year	μ	σ^2	θ	μ	σ^2	θ	μ	σ^2	θ
2003	9.060	0.631	9,066	9.018	0.608	7,664	9.741	0.197	2.071
2004	9.215	0.581	8,552	9.241	0.508	6,541	9.780	0.232	0.222
2005	9.277	0.573	8,873	9.283	0.515	6,977	9.834	0.229	0.270
2006	9.314	0.578	9,383	9.284	0.543	7,868	9.891	0.211	0.591
2007	9.382	0.681	10,028	9.388	0.601	7,903	9.950	0.268	0.162
2008	9.439	0.689	10,898	9.423	0.624	8,755	10.017	0.264	0.190
2009	9.444	0.704	10,641	9.431	0.631	8,685	10.020	0.269	0.200
2010	9.482	0.681	10,617	9.453	0.621	8,746	10.034	0.270	0.201

Year	Criterion *S*	Criterion *S*	Criterion *S*
2003	108,437.01	133,320.79	248,331.74
2004	146,509.34	248,438.78	281,541.41
2005	137,422.05	231,978.79	311,008.23
2006	149,144.68	216,373.24	325,055.67
2007	198,670.74	366,202.87	370,373.62
2008	206,698.93	357,668.48	391,346.02
2009	193,559.55	335,999.20	359,736.37
2010	210,434.01	235,483.68	389,551.44

Source: Own research

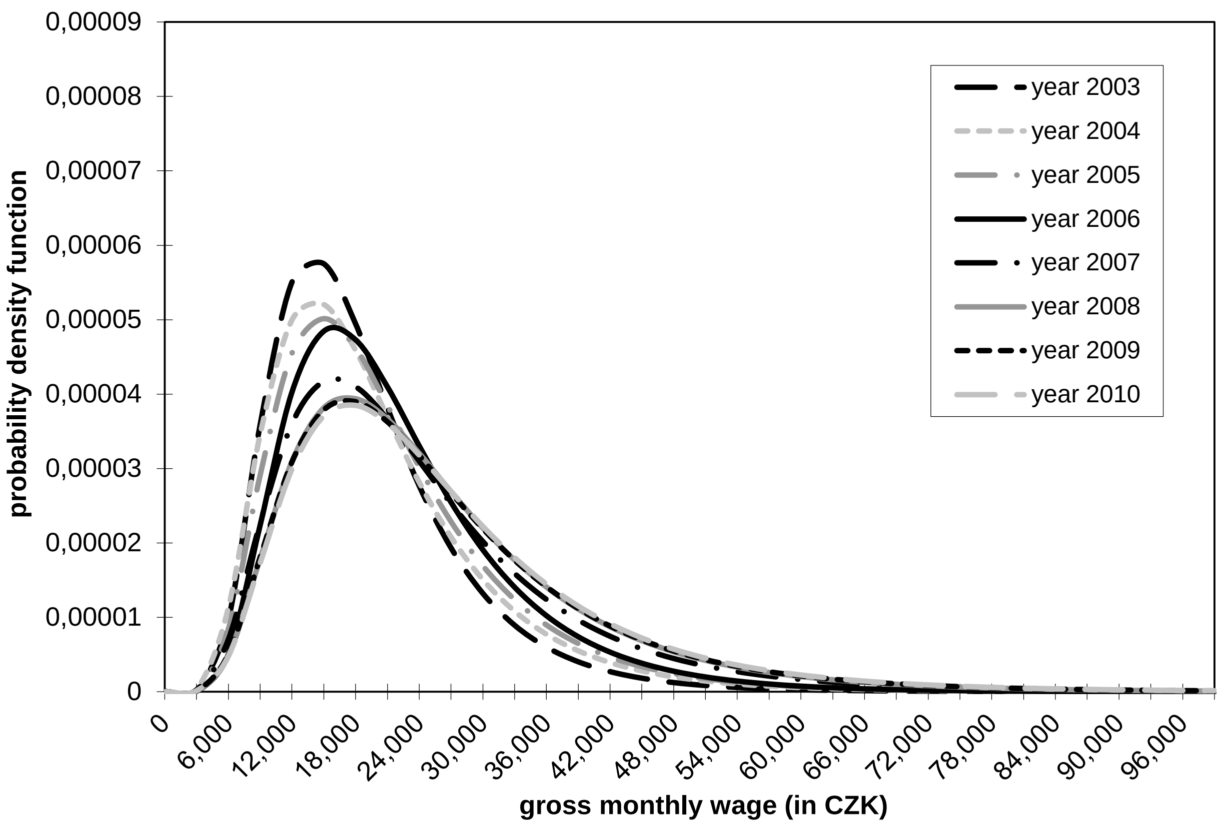

Fig. 22 Development of the probability density function of three-parameter lognormal curves with parameters estimated using the method of TL-moments

Source: Own research

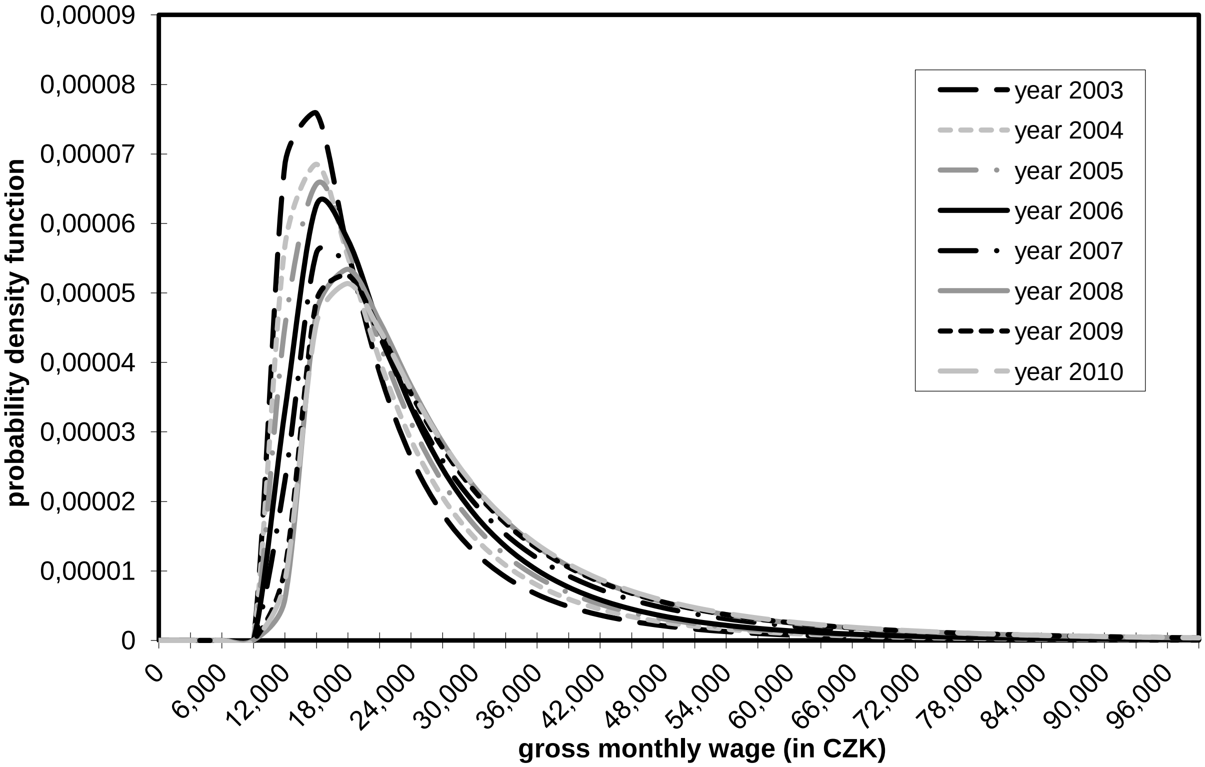

Fig. 23 Development of the probability density function of three-parameter lognormal curves with parameters estimated using the method of L-moments

Source: Own research

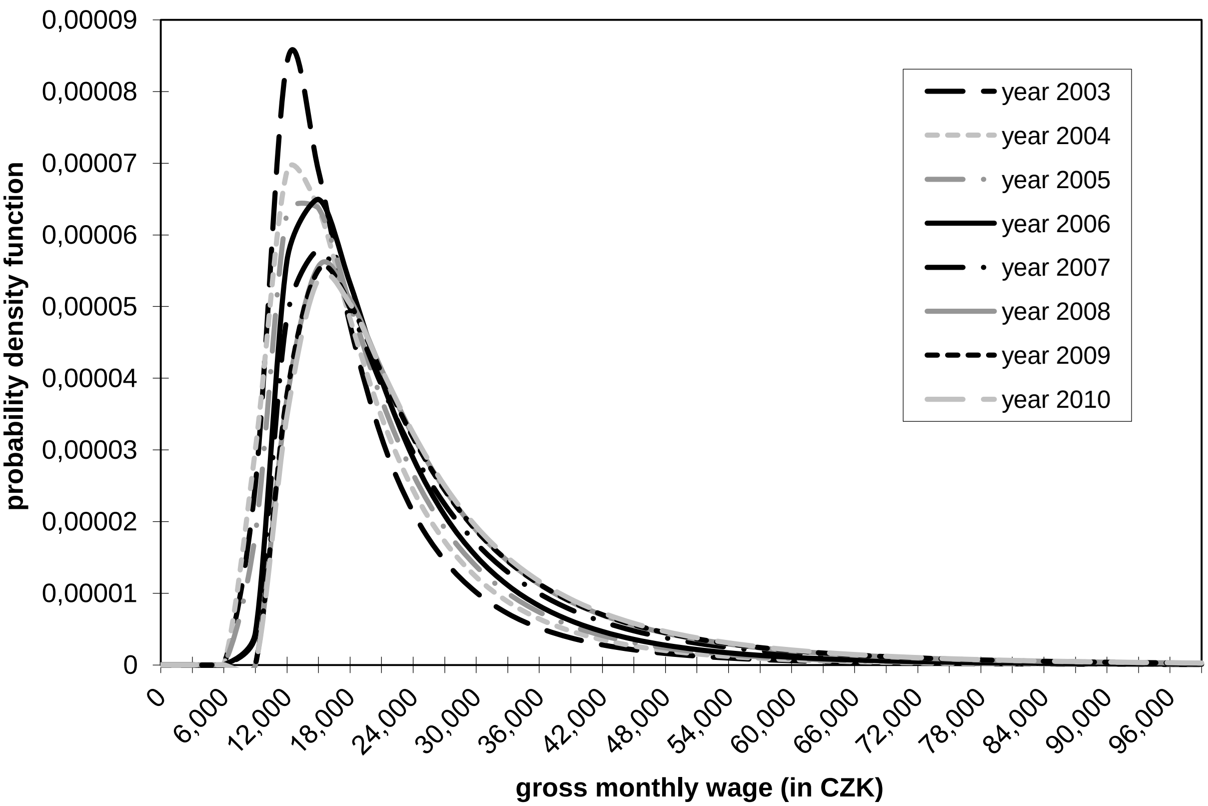

Fig. 24 Development of the probability density function of three-parameter lognormal curves with parameters estimated using the maximum likelihood method

Source: Own research

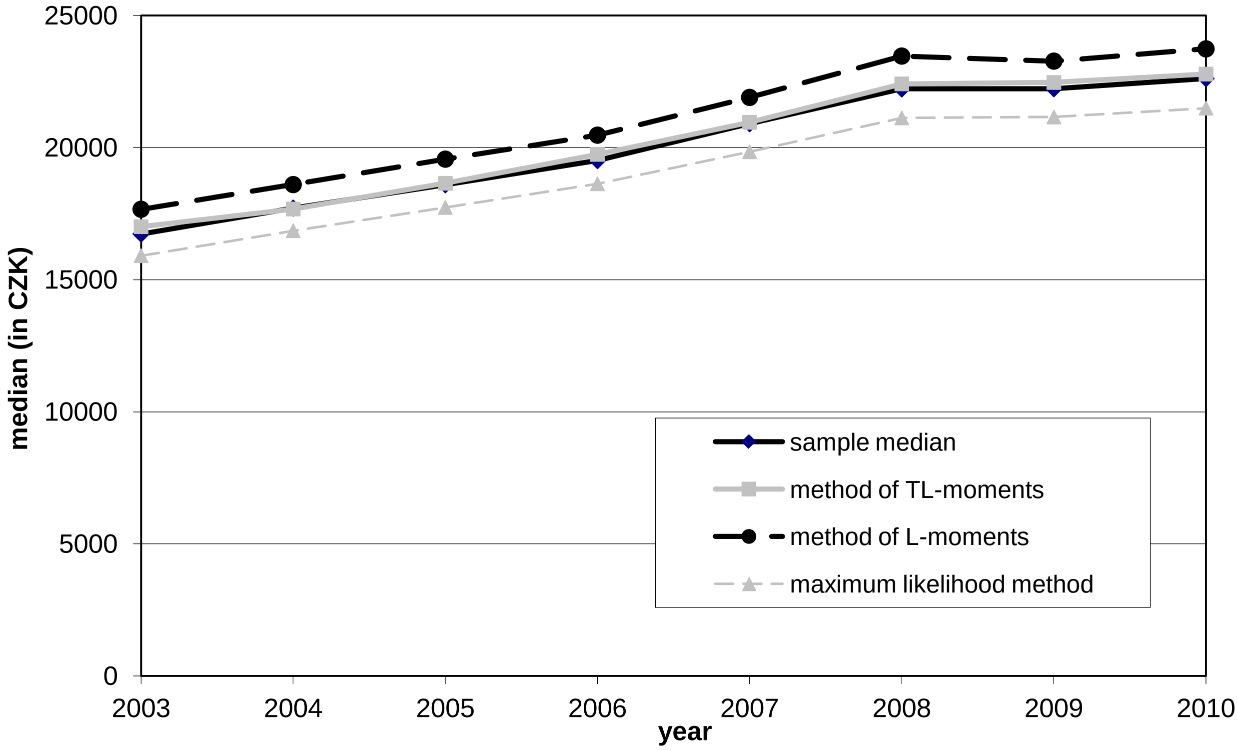

Fig. 25 Development of the sample and theoretical median of three-parameter lognormal curves with parameters estimated using the three methods of parameter estimation

Source: Own research

Table XVI shows parameter estimations obtained using the three methods and the value of the criterion (105) for the total wage distribution in the Czech Republic, giving an approximate description of research outcomes for all 328 wage distributions. We found out that the method of TL-moments provided the most accurate results in almost all, with minor exceptions, wage distribution cases, the deviations having occurred mainly at both ends of the wage distribution due to extreme open intervals of an interval frequency distribution. Table XVI indicates that for the total wage distribution set for the whole Czech Republic in 2003–2010, the method of TL-moments always yields the most accurate output in terms of the *S* criterion. As for the research of all 328 wage distributions, the second most accurate results were produced by the method of L-moments, the deviations having occurred again at both ends of the distribution in particular. The latter method brought the second most accurate results in terms of all total wage distribution data sets over the period 2003–2010. In the majority of cases, the maximum likelihood method was the third most accurate approach. (For all cases, see Table XVI.)

Figs. 22–24 present the development of the probability density function of three-parameter lognormal curves with the parameters estimated employing the methods of TL-moments, L-moments and maximum likelihood, models of the total wage distribution for all employees of the Czech Republic being examined over the period 2003–2010 again. In comparison to the results obtained by the analysis of income distribution, we can see that the shapes of lognormal curves with the parameters estimated using L-moments and maximum likelihood methods (Figs. 23 and 24) are similar to each other, differing greatly, however, from the shape of three-parameter lognormal curves with the parameters estimated by the method of TL-moments (Fig. 22).

Fig. 25 also informs about the accuracy of the examined methods of point parameter estimation. The figure shows the development of the sample median of gross monthly wage for the total set of all employees of the Czech Republic in the period 2003–2010 as well as the development of the respective theoretical median of three-parameter lognormal model curves with the parameters estimated by the three methods. It is observable from this figure that the curve following the course of the theoretical median of a three-parameter lognormal distribution with the parameters estimated using the method of TL-moments adheres the most to the curve showing the development of the sample median. The other two curves articulating the development of the theoretical median of three-parameter lognormal curves with the parameters estimated by L-moments and by maximum likelihood methods are relatively distant from the course of the sample median of the wage distribution.

Figs. 26 and 27 indicate the values of *S* criterion of 2010 wage distributions in terms of job category and five-year age intervals, respectively. High accuracy of the method of TL-moments in comparison to the other two methods of point parameter estimation is evident from the two figures, too.

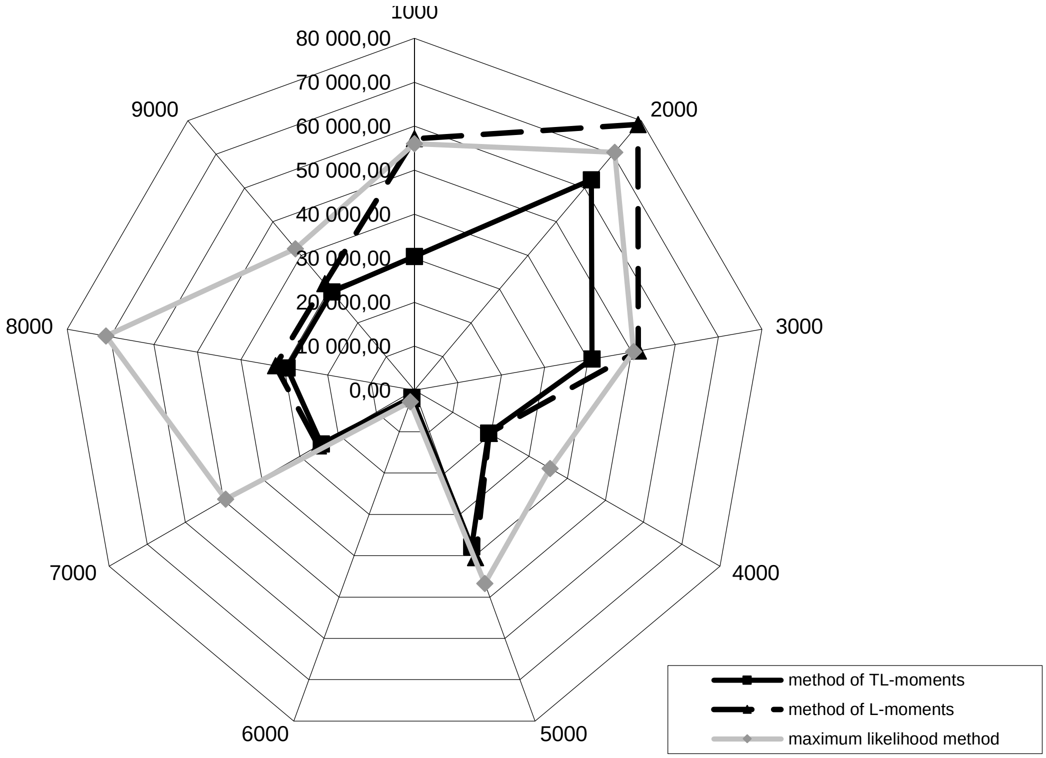

Fig. 26 Values of *S* criterion for three-parameter lognormal model curves with parameters estimated by methods of point parameter estimation (broken down by job category codes) – year 2010

Source: Own research

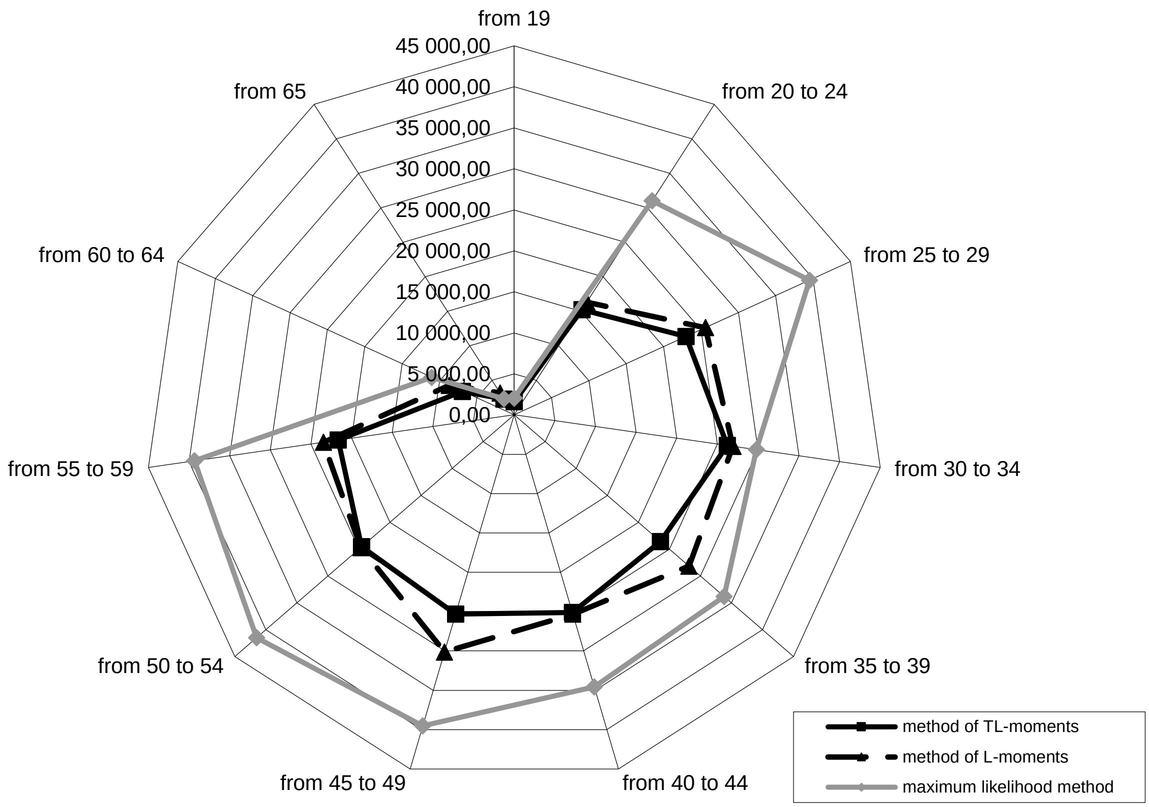

Fig. 27 Values of *S* criterion for three-parameter lognormal model curves with parameters estimated by methods of point parameter estimation (broken down by age-year intervals) – year 2010

Source: Own research

V. The Conclusion

A relatively new class of moment characteristics of the probability distribution has been introduced in this paper. The probability distribution characteristics of the location (level), variability, skewness and kurtosis have been constructed using L-moments and their robust extension – TL-moments method, the former (as an alternative to classical moments of probability distributions) lacking some robust features that are typical for the latter.

Sample TL-moments are linear combinations of sample order statistics assigning zero weight to a predetermined number of sample outliers. They are unbiased estimates of the corresponding TL-moments of probability distributions. The efficiency of TL-statistics depends on the choice of α – $l_1^{(0)}, l_1^{(1)}, l_1^{(2)}$, for instance, having the smallest variance (the highest efficiency) among other estimations for random samples of normal, logistic and double exponential distributions. Some theoretical and practical aspects of TL-moments need to be further researched anyway.

The accuracy of TL-moments method was compared to that of L-moments and the maximum likelihood method. Higher accuracy of the former approach in comparison to that of the latter two methods has been proved by examining 168 income and 328 wage distribution data sets. Advantages of L-moments over the maximum likelihood method have been demonstrated by the present study as well. Two criteria for tackling income and wage distributions, respectively – namely the χ^2 criterion and the sum of all absolute deviations of the observed and theoretical frequencies for all intervals – have been employed. The χ^2 criterion values have always resulted in rejection of the null hypothesis about the supposed shape of the distribution due to large sample sizes typical for income and wage distribution at any significance level.

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doc. Ing. Diana Bílková, Dr. was born in May 1969 in Jilemnice, the Czech Republic. She graduated in statistics at the Faculty of Informatics and Statistics of the University of Economics, Prague (1992). Since then she has been teaching at the Department of Statistics and Probability of the same UE faculty. On receiving the doctor's degree in statistics (1996), she was appointed associate professor in 2013. In her research, she focuses on the theory of probability and mathematical statistics, particularly modeling, development analysis and predictions of the wage and income distribution. She has authored or co-authored more than a hundred of research papers both in Czech and foreign journals, compiled university textbooks, reviews and external examiner's reports. She has dozens of citation responses in domestic and foreign publications, those indexed on the Web of Science representing approximately a third of them. She participated in eight research projects, being a research team leader in two of them. She is a member of the Czech Statistical Society and a committee member of "International Days of Statistics and Economics" conference monitored in the "Conference Proceedings Citation Index". She has successfully supervised numerous statistics graduate theses.