Presenting a Mathematical Model for Joint Production and Purchasing in a Multi-Product Problem and Determining the Optimal Order Value under Practical Constraints

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Abstract: Production and economic order is one of the most important topics in problems of production and inventory control, and it has long been of interest to researchers. Economic and profitable production requires that a comprehensive and detailed plan is taken and implemented for all stages of production. In the manufacturing companies that are producing a group of products, it is possible that in some periods, customers' demands exceed the production rate. In such conditions, the companies can response to such demand by more production, purchase, or accepting shortage of backordering. In case of purchase, the products stock in the warehouse at the beginning of programing period. In this study, a model is presented to determine the optimal amount of production and purchase in a multi-product system and with the objective of optimizing the costs of inventory system under the limitations of warehouse space and budget. Also, the start production time and the optimal amount of the shortage are determined. The presented mathematical model is of nonlinear integer type. The metaheuristic algorithm was used to solve the models, and the method of design of the experiments has been used to configure the parameters of the proposed method. At the end, the proper performance of the proposed methods will be proved by some numerical examples and comparing with random search method.

Keywords: Economic order quantity, Production, Purchasing, Genetic algorithm.

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I. Introduction

The classical model of economic order quantity (EOQ) was first invented by Harris [1] in 1915. In 1963, Hadley and Whitin [2] expanded several deterministic models. With regard to deterministic models, Nador [3] took one step forward and studied a model of economic production which held the assumption of the receiving immediate orders as well as permitting inventory shortages, and in this way extended the previous classical models. Years later, Hadley and Whitin [4] studied various inventory models of EOQ and economic production quantity (EPQ) under the conditions of variable demand parameter and inventory shortage. Lewis [5] studied a discount range when the purchase was over than EOQ. Matsuyama [6] examined EOQ model with the assumptions of discount in purchase price or an increase in setup cost. Klein and Chang [7] reviewed the optimal inventory policy to maximize the benefit in EOQ models under different cost functions. Spikas [8] studied the model of EOQ and EPQ assuming constant linear costs and overdue orders. Pentico and Drake [9] studied the deterministic EPQ model with partial backordering. Lee et al. [10] extended an EPQ model based on the strategy of delay in delivery time in which the inventory shortage of backordered type was considered. Baren [11] presented a simple way in order to achieve the optimum amount of production or purchase and the shortage level with linear and constant costs. Drake et al. [12] presented a method for planning the order size in two-stage inventory system in which the programing of final product was done by using EPQ model and taking into account the shortage issue. Shang and Baren [13] introduced an analytical method for EPQ and EOQ models with linear and fixed shortage costs in order to determine and guarantee the optimal answers. Taleizadeh et al. [14] also solved a nonlinear integer programming model for multi-product inventory control problems with the use of harmony algorithm.

This paper involves with a Combinatory model of production and purchase for the problem of a multi-product-multi-period by considering the allocation of products to machinery and warehouse space limitation. The objective of this model is reducing the inventory costs.

II. PROBLEM DEFINITIONS

In order to control and comprehensive planning of an EOQ or EPQ system, the different conditions should be considered to minimize the delay in supplying orders, avoid additional costs, and achieve high efficiency. In this study, we studied a system that results from integration of EOQ with EPQ. This means that we consider a production system that has also a purchase system to respond the customers' demands. We extend a mixed integer nonlinear programming model for the problem of multi-product multi-period in production condition with purchase and assuming the acceptance of backordered shortage. To make it closer to real world, we also considered warehouse constraints and allocation of products to machines. The objective of the model is to determine the optimal quantity of production, purchase, and also determine the inventory at the time of the start of production, the amount of the shortage and the allocation of products to machines in such a way that the costs related to the production and purchase be minimized in a certain upcoming programming.

A. Assumptions

The assumptions of this mode are:

- The replenishment is momentary and in the start time of each period,
- Demand of all products in each period are constant and independent from other periods,
- Production rate of all products in each period are constant and independent from other periods,
- Each planning horizon includes some periods,
- Backorder is allowed for each production,
- Total budget for purchasing and producing in the start of each period is limited,
- Replenishment has been done it constant time intervals,
- In each period only one purchase can be done in the start of period,
- Number of production machines considered as infinitive and all products can be produced by all machines,

B. Parameters

The mathematical model parameters are:

D_{it} : Demand rate of i[st] product in period t

F : Maximum storage capacity

h_{it} : Holding cost of i[st] product in period t

A'_{it} : Setup cost of i[st] product in period t

C'_{it} : Production cost of i[st] product in period t

λ_{it} : Maximum inventory level of i[st] product in period t

β : Infinite positive large number

THC : Total inventory cost

TOC : Total ordering and setup cost

TMC : Total machinery cost

P_{it} : Production rate of i[st] product in period t

w_i : Coefficient of base product valume

A_{it} : Ordering cost of i[st] product in period t

C_{it} : Purchasing cost of i[st] product in period t

C'''_{itk} : Cost of using machine k for producing i[st] product in period t

π_{it} : Backorder cost of i[st] product in period t

S_i : Setup time of production i

TPC : Total purchasing and production cost

TBC : Total backorder cost

T^Q_{it} : Interval between purchasing and using production i in time t until the level of inventory is in level R_{it}

T^P_{it} : Interval between the start time of producing production i in time t until stop time of production

T^D_{it} : Interval between the stop time of producing production i in time t until the end of period

$i = 1,2,\dots,N$

$t = 1,2,\dots,T$

C. Decision variables

R_{it} : Inventory value of i[st] product is start of time period t

y_{it} : Binary variable, if in production time the value of inventory is positive consider 1, else consider 0

Q_{it} : Purchasing value of i[st] product is start of time period t

Q'_{it} : production value of i[st] product in time period t

b_{it} : backorder value of i[st] product is start of time period t

M_{ikt} : Binary variable, if i[st] product produce by k[st] machine in time t consider 1, else consider 0

D. Mathematical model:

$$\min TC = \sum_{i=1}^{N}\sum_{t=1}^{T-1}(C_{it}Q_{it} + C'_{it}Q'_{it})\frac{D_{it}}{Q_{it}+Q'_{it}+b_{it}} + (A_{it}+A'_{it})\frac{D_{it}}{Q_{it}+Q'_{it}+b_{it}} + h_{it}\frac{D_{it}}{Q_{it}+Q'_{it}+b_{it}}\{$$

$$(\frac{(Q_{it}+R_{it})\left(T_{it}^{Q}-\frac{b_{it}}{D_{it}}\right)}{2} + \frac{\left(2R_{it}+T_{it}^{P}(P_{it}-D_{it})\right)T_{it}^{P}}{2} + \frac{\left(R_{it}+T_{it}^{P}(P_{it}-D_{it})\right)\left(T_{it}^{D}-\frac{b_{it}}{D_{it}}\right)}{2})y_{it}$$

$$+(\frac{Q_{it}\left(T_{it}^{Q}+\frac{R_{it}}{D_{it}}\right)}{2} + \frac{W_{it}^{P}\left(T_{it}^{P}(P_{it}-D_{it})+R_{it}\right)\left(T_{it}^{P}+\frac{R_{it}}{(P_{it}-D_{it})}\right)}{2} + \frac{W_{it}^{D}\left(\left(R_{it}+T_{it}^{P}(P_{it}-D_{it})\right)\left(T_{it}^{D}-\frac{b_{it}}{D_{it}}\right)\right)}{2})(1-y_{it})\}$$

$$+\pi_{it}\frac{D_{it}}{Q_{it}+Q'_{it}+b_{it}}\{(\frac{b_{it}\left(T_{it}^{D}-\frac{T_{it}^{P}(P_{it}-D_{it})+R_{it}}{D_{it}}\right)}{2})y_{it} +$$

$$(\frac{W_{it}^{P}R_{it}^{2}}{2D_{it}} + \frac{W_{it}^{P}R_{it}^{2}}{2(P_{it}-D_{it})} + \frac{b_{it}(T_{it}^{D}-\frac{T_{it}^{P}(P_{it}-D_{it})+R_{it}}{D_{it}})}{2})(1-y_{it})\} + \sum_{i=1}^{N}\sum_{k=1}^{K}\sum_{t=1}^{T}W_{it}^{P}C''_{k}M_{ikt}$$

Subject to:

$$T_{it}^{Q} = \frac{Q_{it}-R_{it}}{D_{it}}, \ \forall i \in I, t \in T$$

$$T_{it}^{P} = \frac{Q'_{it}}{P_{it}}, \ \forall i \in I, t \in T$$

$$T_{it}^{D} = \frac{R_{it}+T_{it}^{P}(P_{it}-D_{it})+b_{it}}{D_{it}}, \ \forall i \in I, t \in T$$

$$R_{it} + \varepsilon \le y_{it}.\beta, \ \forall i \in I, t \in T$$

$$R_{it} \ge -(1-y_{it}).\beta, \ \forall i \in I, t \in T$$

$$T_{it}^{P} \le W_{it}^{P}.\beta, \ \forall i \in I, t \in T$$

$$T_{it}^{D} \le W_{it}^{D}.\beta, \ \forall i \in I, t \in T$$

$$T_{it}^{S} = T_{it}^{Q} + T_{it}^{P} + T_{it}^{D}, \ \forall i \in I, t \in T$$

$$T_{it}^{S} = T'_{total}, \ \ \forall i \in I, t \in T$$

$$\sum_{i=1}^{N} M_{itk}(T_{it}^{P} + S_{i}) \le T'_{total}, \ \ t \in T, k \in K$$

$$\sum_{k=1}^{K} M_{itk} \leq 1, \ \ i \in N, t \in T$$

$$M_{itk} \leq T_{it}^{p}.\beta, \ \ i \in N, k \in K, t \in T$$

$$\sum_{i=1}^{N} w_i \lambda_{it} \leq F, \ \ \forall t \in T$$

$$\lambda_{it} = Z_{it} Q_{it} + (1 - Z_{it})(R_{it} + T_{it}^{P}(P_{it} - D_{it})), \ \ \forall i \in I, t \in T$$

$$\lambda_{it} \geq Q_{it}, \ \ \forall i \in I, t \in T$$

$$\lambda_{it} \geq R_{it} + T_{it}^{P}(P_{it} - D_{it}), \ \ \forall i \in I, t \in T$$

$$Q_{it}, Q'_{it} \geq 0, \ \ y_{it}, Z_{it} \in \{0,1\}, \forall i \in I, t \in T, \ i = \{1,2,\ldots,N\}, \ t = \{0,1,2,\ldots,T\}$$

III. Solving method

In this section, we use Genetic algorithm to solve the proposed integer nonlinear programming problem.

A. *Genetic Algorithm*

Genetic algorithm was first introduced by Holland [15] at Michigan University and evolution strategies and evolutionary programing developed by Rechenberg, Schwefel, Fogel, and Koza are among evolutionary calculating methods.

B. *Solution encoding*

The general form of related responses to the proposed mathematical model irrespective of using any solution method includes four matrixes (Figure 1).

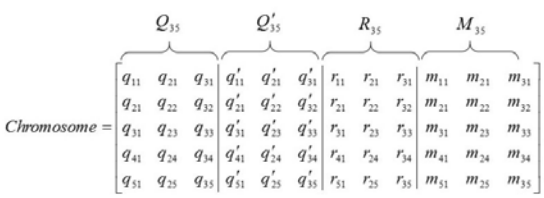

Fig.1: Solution encoding.

C. *Random search*

A random search (RS) method to solve the proposed model could be an upper bound (for minimizing problems) for other solving methods. In other words, the proof for intelligent function of metaheuristic algorithms can be shown by comparing them with an RS. So that these algorithms must always work stronger than an RS. Thus to prove the proposed, an RS is presented.

D. *Parameters tuning*

In this article, we used response surface methodology to tune the problem parameters. Table 1 presents the parameters of Genetic algorithm and their optimal values.

Table 1.

Search interval and optimal levels of input variables of Genetic algorithm.

parameter	**Lower bound**	**Upper bound**	**Opoptimal value**
npop	100	200	162
p_C	0.6	0.9	0.7542770
p_M	0.1	0.3	0.2013418

IV. Numerical example

Assume a numerical example with 10 products in 3 time periods. There are 4 machines to produce them that cost of using each (C″) would be 800, 1100, 900, 1000 and maximum warehouse capacity for inventory is 10000 units. Table 5.5 presents demands rate, production capacity, purchase cost, production cost, and holding cost. Also Table 2 presents shortage cost, ordering cost, setup cost, and production size.

The solutions to 10 solved problems with Genetic algorithm and RS are presented in Table 3. Also, to compare the solutions obtained by two methods, we used the Chi-squared test. Based on Table 4, both solutions was equal indicating the accuracy of the proposed algorithm.

TABLE 2.

THE INPUT PARAMETERS OF PROBLEM FOR NUMERIC EXAMPLE.

i	*D*			*P*			*C*			*C'*			*h*		
	t=1	*t=2*	*t=3*	*t=1*	*t=2*	*t=3*	*t=1*	*t=2*	*t=3*	*t=1*	*t=2*	*t=3*	*t=1*	*t=2*	*t=3*
1	18.2	19.1	11.2	259	213	154	53	52	77	55	99	50	40	20	30
2	19.2	10.3	10.9	215	590	195	89	23	56	89	31	94	38	6	49
3	12.8	15.5	19.6	204	200	292	79	73	50	54	70	63	32	15	16
4	19.7	11.5	19.8	219	170	248	67	58	90	90	72	96	20	44	20
5	19.6	14.9	18.0	278	233	215	65	76	58	59	63	57	43	19	48
6	11.4	14.2	19.2	173	198	264	80	63	83	56	94	79	24	18	20
7	18.0	19.6	16.6	257	277	208	85	88	72	78	57	93	35	29	24
8	10.3	18.5	19.4	178	258	227	54	61	96	81	67	76	44	33	32
9	16.8	17.6	17.5	197	236	272	57	92	77	70	53	62	47	21	41
10	13.9	16.6	11.7	186	233	155	100	53	72	56	59	62	40	25	33

TABLE 3.

THE OPTIMAL SOLUTION VALUES OF GENETIC ALGORITHM AND RANDOM SEARCH FOR 10 PROBLEMS.

F	GA	RS
19000	86428.8882	825622.89
17000	88410.9759	699583.8
15000	89650.4522	523137.6
13000	90621.0234	802895.1
10000	92353.6634	763405.2
7000	93072.9085	867389.7
5000	93457.19845	890702.8
3000	94270.3085	753096.34
1000	95045.5867	844757.5
100	99280.871	393777.8

TABLE 4.

THE CHI-SQUARED RESULTS FOR TESTING THE SIGNIFICANT DIFFERENCE BETWEEN THE RESULTS OF GENETIC ALGORITHM AND RANDOM SEARCH.

Two-sample T for RS vs GA				
	N	Mean	StDev	SE Mean
RS	10	736437	159974	50588
GA	10	92259	3674	1162

Difference = mu (RS) - mu (GA)
Estimate for difference: 644178
95% CI for difference: (529709, 758646)
T-Test of difference = 0 (vs not =): T-Value = 12.73 P-Value = 0.000
DF = 9

V. CONCLUSION AND FURTHER STUDIES

A. *Conclusion*

In this research, a mathematical integer nonlinear programing model was developed for the control of inventory of a multi-product multi-period situation to decide on production with purchase and allocation of products to machines. To make the model more realistic, the limitation in warehouse space and order size were considered in the model, too. The objective was to minimize the costs of inventory system with respect to a determined programing through deciding on optimal purchase quantity, production, shortage, and level of inventory that production has started from there in each period. A metaheuristic algorithm has been used to solve the model and random search method to verify the results. According to the results presented in chapter 5 (comparing with random search method), the intelligent performance of proposed metaheuristic algorithms was proved.

B. *Further studies*

- As in the real world and with respect to order size, discount is possible, it is suggested that in the model, the purchase cost is considered under the discount condition. This discount could be general or incremental.
- It is suggested that shortage is considered as a combination of backordering shortage and lost sale.

- It is suggested that the model be solved by modifying (simplifying) it through precise methods like sequential unconstrained minimization technique (SUMT).

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