On the Split (s,t)-Padovan and (s,t)-Perrin Quaternions

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Abstract—In this paper we consider the generalization of Padovan and Perrin quaternions. We define the split (s,t)-Padovan and (s,t)-Perrin quaternions which generalize Padovan and Perrin quaternions. We derive the Binet-like formulas for the split (s,t)-Padovan and (s,t)-Perrin quaternions. We establish their generating functions. Also, we obtain certain binomial sums regarding the split (s,t)-Padovan and (s,t)-Perrin quaternions.

Keywords—Padovan quaternion, Perrin quaternion, Binet-like formula, Generating function, Binomial sum.

I. Introduction

The split quaternions were defined by James Cockle in 1849. The quaternions were defined by Hamilton in 1943 as an extension to the complex numbers. They are formed a four dimensional real vector space with a multiplicative operation. They have played a significant role in physical science, differential geometry, analysis and synthesis of mechanism and mechines, theory of relativity and others. Unlike the quaternion algebra, the split quaternions contain zero divisors, nilpotent elements and non-trivial idempotents. A split quaternion is defined by

$$q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$$

where q_0, q_1, q_2 and q_3 are real numbers and $e_0 = 1$, $e_1 = i$, $e_2 = j$ and $e_3 = k$ are the standard basis in $\mathbb{R}^4$. Then we can write

$$q = S_q + V_q$$

where $S_q = q_0e_0$ and $V_q = q_1e_1 + q_2e_2 + q_3e_3$. S_q is called the scalar part of the split quaternion q and V_q is called the vector part of the split quaternion q. The split quaternion multiplication is defined using the rules;

$$e_1^2 = -1, \quad e_0^2 = e_2^2 = e_3^2 = 1$$

$$e_1e_2 = -e_2e_1 = e_3$$

$$e_2e_3 = -e_3e_2 = -e_1$$

and

$$e_3e_1 = -e_1e_3 = e_2.$$

This algebra is associative and non-commutative . Let $q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$ and $p = p_0e_0 + p_1e_1 + p_2e_2 + p_3e_3$ be any two split quaternions. Then the addition and subtraction of the split quaternions is

$$q \mp p = (q_0 \mp p_0)e_0 + (q_1 \mp p_1)e_1 + (q_2 \mp p_2)e_2 + (q_3 \mp p_3)e_3$$

and multiplication of the split quaternions is

$$\begin{aligned} qp &= (q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3)(p_0e_0 + p_1e_1 + p_2e_2 + p_3e_3) \\ &= (q_0p_0 - q_1p_1 + q_2p_2 + q_3p_3)e_0 \\ &+ (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)e_1 \\ &+ (q_0p_2 + q_2p_0 - q_1p_3 + q_3p_1)e_2 \\ &+ (q_0p_3 + q_3p_0 + q_1p_2 - q_2p_1)e_3 \end{aligned}$$

and for $k \in \mathbb{R}$ the multiplication by scalar is

$$kq = kq_0e_0 + kq_1e_1 + kq_2e_2 + kq_3e_3.$$

The basic operations on the two split quaternions given above can also be seen in [1] and [7].

Special number sequences have play important role in mathematics and applied sciences. Moreover, some special number sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Padovan and Perrin sequences have many applications in art, music, photography, architecture, painting, engineering, geometry and others. It is well-known that the term golden ratio is defined the ratio of two consecutive Fibonacci numbers converges to

$$\frac{1+\sqrt{5}}{2} \approx 1.618034.$$

The golden ratio has many applications in engineering, physics, architecture, arts and other. In similar way, the ratio of two consecutive Padovan or Perrin numbers converges to

$$\sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{23}{3}}} \approx 1.324718,$$

that is called as "plastic ratio". The plastic ratio (number) was first defined by Gerard Cordonnier in 1924. He described applications to architecture and illustrated the use of the plastic number in many buildings. Furthermore, the plastic number is the unique real root of the equation

$$x^3 - x - 1 = 0,$$

the characteristic equation of Padovan number sequences. (see [4], [6], [9]). The Padovan sequence $\{P_n\}_{n\geq 0}$ is defined by the initial values $P_0 = P_1 = P_2 = 1$ and the recurrence relation

$$P_{n+3} = P_{n+1} + P_n, \qquad \text{for all } n \geq 0. \tag{1}$$

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First few terms of this sequence are $1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28$. The Perrin sequence $\{R_n\}_{n\geq 0}$ is defined by the initial values $R_0 = 3, R_1 = 0$ and $R_2 = 2$ and the recurrence relation

$$R_{n+3} = R_{n+1} + R_n, \qquad \text{for all } n \geq 0. \tag{2}$$

First few terms of Perrin sequence are $3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29$. Padovan and Perrin sequence can be found in [2], [6], [8], [9].

A generalization of the Padovan sequence $\{P_n\}_{n\geq 0}$, which is called the (s,t)-Padovan sequence, say $\{\mathcal{P}_n(s,t)\}_{n\geq 0}$ is defined by the following recurrence relation for $n \geq 0$ and any integer numbers $s > 0$ and $t \neq 0$ such that $27t^2 - 4s^3 \neq 0$:

$$\mathcal{P}_{n+3}(s,t) = s\mathcal{P}_{n+1}(s,t) + t\mathcal{P}_n(s,t), \tag{3}$$

where $\mathcal{P}_0(s,t) = 0, \quad \mathcal{P}_1(s,t) = 1$ and $\mathcal{P}_2(s,t) = 0$. To simplify notation, we take $\mathcal{P}_n(s,t) = \mathcal{P}_n$.

A generalization of the Perrin sequence $\{R_n\}_{n\geq 0}$, which is called the (s,t)-Perrin sequence, say $\{\mathcal{R}_n(s,t)\}_{n\geq 0}$ is defined by the following recurrence relation for $n \geq 0$ and any integer numbers $s > 0$ and $t \neq 0$ such that $27t^2 - 4s^3 \neq 0$:

$$\mathcal{R}_{n+3}(s,t) = s\mathcal{R}_{n+1}(s,t) + t\mathcal{R}_n(s,t), \tag{4}$$

where $\mathcal{R}_0(s,t) = 3, \quad \mathcal{R}_1(s,t) = 0$ and $\mathcal{R}_2(s,t) = 2s$. To simplify notation, take $\mathcal{R}_n(s,t) = \mathcal{R}_n$. The (s,t)-Padovan and (s,t)-Perrin sequences were investigated in [2].

For every $x \in \mathbb{N}$, one can write the Binet-like formulas for the (s,t)-Padovan and (s,t)-Perrin sequences as the form

$$\mathcal{P}_n = a\alpha^n + b\beta^n + c\gamma^n \tag{5}$$

and

$$\mathcal{R}_n = \alpha^n + \beta^n + \gamma^n \tag{6}$$

where α, β and γ are the roots of the characteristic equation

$$x^3 - sx - t = 0 \tag{7}$$

associated with (1) and (2), where

$$a = \frac{(\beta-1)(\gamma-1)}{(\alpha-\beta)(\alpha-\gamma)}, \quad b = \frac{(\alpha-1)(\gamma-1)}{(\beta-\alpha)(\beta-\gamma)},$$

$$c = \frac{(\alpha-1)(\beta-1)}{(\alpha-\gamma)(\beta-\gamma)}.$$

The Binet-like formulas for the $(s,t)-$Padovan and $(s,t)-$Perrin sequences were given in [2]. In the present work we define the nth split Padovan and Perrin quaternions by the formulas

$$\mathcal{S}P_n = P_n e_0 + P_{n+1}e_1 + P_{n+2}e_2 + P_{n+3}e_3 \tag{8}$$

and

$$\mathcal{S}R_n = R_n e_0 + R_{n+1}e_1 + R_{n+2}e_2 + R_{n+3}e_3 \tag{9}$$

where P_n and R_n are the nth Padovan and Perrin number.

II. Split $(s,t)-$Padovan and $(s,t)-$ Perrin Quaternions

The (p,q)-Fibonacci quaternions were defined and studied in [5]. As generalization of the Padovan and Perrin, the (s,t)-Padovan and (s,t)-Perrin quaternions were defined and investigated in [3]. In this work we consider their split cases. In this section, we define two new split quaternions that are split $(s,t)-$Padovan and $(s,t)-$Perrin quaternions. Then, we give their Binet-like formulas, generating functions and certain binomial sums.

Definition 1. *The split $(s,t)-$Padovan quaternion sequence $\{\mathcal{SP}_n(s,t)\}_{n\geq 0}$ is defined by*

$$\mathcal{SP}_n(s,t) = \mathcal{P}_n e_0 + \mathcal{P}_{n+1}e_1 + \mathcal{P}_{n+2}e_2 + \mathcal{P}_{n+3}e_3 \tag{10}$$

where $\mathcal{P}_n$ is the nth $(s,t)-$Padovan number. To simplify notation, we take $\mathcal{SP}_n(s,t) = \mathcal{SP}_n$.

Definition 2. *The split $(s,t)-$Perrin quaternion sequence $\{\mathcal{SR}_n(s,t)\}_{n\geq 0}$ is defined by*

$$\mathcal{SR}_n(s,t) = \mathcal{R}_n e_0 + \mathcal{R}_{n+1}e_1 + \mathcal{R}_{n+2}e_2 + \mathcal{R}_{n+3}e_3 \tag{11}$$

where $\mathcal{R}_n$ is the nth $(s,t)-$Perrin number. To simplify notation, we take $\mathcal{SR}_n(s,t) = \mathcal{SR}_n$.

Theorem 3 (Binet-like formula)**.** *The Binet-like formulas for the nth split $(s,t)-$Padovan quaternion is*

$$\mathcal{SP}_n = a\hat{\alpha}\alpha^n + b\hat{\beta}\beta^n + c\hat{\gamma}\gamma^n, \qquad n \geqslant 0 \tag{12}$$

where

$$\hat{\alpha} = e_0 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3,$$

$$\hat{\beta} = e_0 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3,$$

and

$$\hat{\gamma} = e_0 + \gamma e_1 + \gamma^2 e_2 + \gamma^3 e_3.$$

Proof: From the definition of nth split $(s,t)-$Padovan quaternion $\mathcal{SP}_n$ in (10) and Binet-like formula for the nth $(s,t)-$Padovan number $\mathcal{P}_n$, we write

$$\begin{aligned}
\mathcal{SP}_n &= \mathcal{P}_n e_0 + \mathcal{P}_{n+1}e_1 + \mathcal{P}_{n+2}e_2 + \mathcal{P}_{n+3}e_3 \\
&= (a\alpha^n + b\beta^n + c\gamma^n)e_0 \\
&+ (a\alpha^{n+1} + b\beta^{n+1} + c\gamma^{n+1})e_1 \\
&+ (a\alpha^{n+2} + b\beta^{n+2} + c\gamma^{n+2})e_2 \\
&+ (a\alpha^{n+3} + b\beta^{n+3} + c\gamma^{n+3})e_3 \\
&= a(e_0 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3)\alpha^n \\
&+ b(e_0 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3)\beta^n \\
&+ c(e_0 + \gamma e_1 + \gamma^2 e_2 + \gamma^3 e_3)\gamma^n \\
&= a\hat{\alpha}\alpha^n + b\hat{\beta}\beta^n + c\hat{\gamma}\gamma^n
\end{aligned}$$

Thus, the proof is completed. ■

Theorem 4 (Binet-like formula)**.** *The Binet-like formula for the nth split $(s,t)-$Perrin quaternion is*

$$\mathcal{SR}_n = \hat{\alpha}\alpha^n + \hat{\beta}\beta^n + \hat{\gamma}\gamma^n, \qquad n \geqslant 0 \tag{13}$$

where

$$\hat{\alpha} = e_0 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3,$$
$$\hat{\beta} = e_0 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3,$$

and

$$\hat{\gamma} = e_0 + \gamma e_1 + \gamma^2 e_2 + \gamma^3 e_3.$$

Proof: From the definition of nth split $(s,t)-$Perrin quaternion $\mathcal{SR}_n$ in (11) and Binet-like formula for the nth $(s,t)-$Perrin number $\mathcal{R}_n$, we write

$$\begin{aligned}\mathcal{SR}_n &= \mathcal{R}_n e_0 + \mathcal{R}_{n+1} e_1 + \mathcal{R}_{n+2} e_2 + \mathcal{R}_{n+3} e_3\\ &= (\alpha^n + \beta^n + \gamma^n) e_0\\ &+ (\alpha^{n+1} + \beta^{n+1} + \gamma^{n+1}) e_1\\ &+ (\alpha^{n+2} + \beta^{n+2} + \gamma^{n+2}) e_2\\ &+ (\alpha^{n+3} + \beta^{n+3} + \gamma^{n+3}) e_3\\ &= (e_0 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3)\alpha^n\\ &+ (e_0 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3)\beta^n\\ &+ (e_0 + \gamma e_1 + \gamma^2 e_2 + \gamma^3 e_3)\gamma^n\\ &= \hat{\alpha}\alpha^n + \hat{\beta}\beta^n + \hat{\gamma}\gamma^n\end{aligned}$$

Thus, the proof is completed. ■

Theorem 5. *The generating function for the nth split $(s,t)-$Padovan quaternion is*

$$\mathcal{G}_{\mathcal{SP}}(x) = \frac{e_1 + se_3 + (e_0 + se_2 + te_3)x + (te_2)x^2}{1 - sx^2 - tx^3}.$$

Proof: Assume that the function

$$\begin{aligned}\mathcal{G}_{\mathcal{SP}}(x) = \sum_{n=0}^{\infty} \mathcal{SP}_n x^n &= \mathcal{SP}_0 + \mathcal{SP}_1 x + \mathcal{SP}_2 x^2\\ &+ \mathcal{SP}_3 x^3 + \ldots + \mathcal{SP}_n x^n + \ldots\end{aligned}$$

be generating function of the split $(s,t)-$Padovan quaternions. Multiply both of side of the equality by the term $-sx^2$ such as

$$\begin{aligned}-sx^2\mathcal{G}_{\mathcal{SP}}(x) &= -s\mathcal{SP}_0 x^2 - s\mathcal{SP}_1 x^3 - s\mathcal{SP}_2 x^4\\ &- s\mathcal{SP}_3 x^5 - \ldots - s\mathcal{SP}_n x^{n+2} - \ldots\end{aligned}$$

and multiply by the term $-tx^3$ such as

$$\begin{aligned}-tx^3\mathcal{G}_{\mathcal{SP}}(x) &= -t\mathcal{SP}_0 x^3 - t\mathcal{SP}_1 x^4 - t\mathcal{SP}_2 x^5\\ &- t\mathcal{SP}_3 x^6 - \ldots - t\mathcal{SP}_n x^{n+3} - \ldots\end{aligned}$$

Then, let $(1 - sx^2 - tx^3)\mathcal{G}_{\mathcal{SP}}(x) = A$. We write

$$\begin{aligned}A =&\mathcal{SP}_0 + \mathcal{SP}_1 x + (\mathcal{SP}_2 - s\mathcal{SP}_0)x^2\\ &+ (\mathcal{SP}_3 - s\mathcal{SP}_1 - t\mathcal{SP}_0)x^3 + \ldots\\ &+ (\mathcal{SP}_n - s\mathcal{SP}_{n-2} - t\mathcal{SP}_{n-3})x^n + \ldots\end{aligned}$$

Now, by using

$$\mathcal{SP}_0 = e_1 + se_3,$$
$$\mathcal{SP}_1 = e_0 + se_2 + te_3,$$
$$\mathcal{SP}_2 = se_1 + te_2 + s^2 e_3,$$

and

$$\mathcal{SP}_n - s\mathcal{SP}_{n-2} - t\mathcal{SP}_{n-3} = 0,$$

we obtain that

$$\mathcal{G}_{\mathcal{SP}}(x) = \frac{e_1 + se_3 + (e_0 + se_2 + te_3)x + (te_2)x^2}{1 - sx^2 - tx^3}.$$

Thus, the proof is completed. ■

Theorem 6. *The generating function of the nth split $(s,t)-$Perrin quaternion is*

$$\begin{aligned}\mathcal{G}_{\mathcal{SR}}(x) &= \frac{3e_0 + 2se_2 + 3te_3 + (2e_1 + 3te_2 + 2s^2 e_3)x}{1 - sx^2 - tx^3}\\ &+ \frac{(-se_0 + 3te_1 + 2ste_3)x^2}{1 - sx^2 - tx^3}\end{aligned}$$

Proof: Let

$$\begin{aligned}\mathcal{G}_{\mathcal{SR}}(x) = \sum_{n=0}^{\infty} \mathcal{SR}_n x^n &= \mathcal{SR}_0 + \mathcal{SR}_1 x + \mathcal{SR}_2 x^2\\ &+ \mathcal{SR}_3 x^3 + \ldots + \mathcal{SR}_n x^n + \ldots\end{aligned}$$

be generating function of the split $(s,t)-$Perrin quaternions. Now multiply both of side of the equality by term $-sx^2$ such as

$$\begin{aligned}-sx^2\mathcal{G}_{\mathcal{SR}}(x) &= -s\mathcal{SR}_0 x^2 - s\mathcal{SR}_1 x^3 - s\mathcal{SR}_2 x^4\\ &- s\mathcal{SR}_3 x^5 - \ldots - s\mathcal{SR}_n x^{n+2} - \ldots\end{aligned}$$

and multiply by $-tx^3$ such as

$$\begin{aligned}-tx^3\mathcal{G}_{\mathcal{SR}}(x) &= -t\mathcal{SR}_0 x^3 - t\mathcal{SR}_1 x^4 - t\mathcal{SR}_2 x^5\\ &- t\mathcal{SR}_3 x^6 - \ldots - t\mathcal{SR}_n x^{n+3} - \ldots.\end{aligned}$$

Then, let $(1 - sx^2 - tx^3)\mathcal{G}_{\mathcal{SR}}(x) = B$ we write

$$\begin{aligned}B &= \mathcal{SR}_0 + \mathcal{SR}_1 x + (\mathcal{SR}_2 - s\mathcal{SR}_0)x^2\\ &+ (\mathcal{SR}_3 - s\mathcal{SR}_1 - t\mathcal{SR}_0)x^3 + \ldots\\ &+ (\mathcal{SR}_n - s\mathcal{SR}_{n-2} - t\mathcal{SR}_{n-3})x^n + \ldots\end{aligned}$$

By using

$$\mathcal{SR}_0 = 3e_0 + 2se_2 + 3te_3,$$
$$\mathcal{SR}_1 = 2se_1 + 3te_2 + 2s^2 e_3,$$
$$\mathcal{SR}_2 = 2se_0 + 3te_1 + 2s^2 e_2 + 5ste_3,$$

and

$$\mathcal{SR}_n - s\mathcal{SR}_{n-2} - t\mathcal{SR}_{n-3} = 0,$$

we obtain that

$$\begin{aligned}\mathcal{G}_{\mathcal{SR}}(x) &= \frac{3e_0 + 2se_2 + 3te_3 + (2e_1 + 3te_2 + 2s^2 e_3)x}{1 - sx^2 - tx^3}\\ &+ \frac{(-se_0 + 3te_1 + 2ste_3)x^2}{1 - sx^2 - tx^3}\end{aligned}$$

This completes the proof. ■

Theorem 7. *Let m be a positive integer. Then,*

$$\sum_{n=0}^{m} \binom{m}{n} s^n t^{m-n} \mathcal{SP}_n = \mathcal{SP}_{3m}.$$

Proof: Applying Binet-like formula (12), let $\sum_{n=0}^{m}\binom{m}{n}s^n t^{m-n}\mathcal{SP}_n = C$. We obtain the identities

$$C = \sum_{n=0}^{m}\binom{m}{n}s^n t^{m-n}(a\hat{\alpha}\alpha^n + b\hat{\beta}\beta^n + c\hat{\gamma}\gamma^n)$$
$$= \sum_{n=0}^{m}\binom{m}{n}(a\hat{\alpha}(s\alpha)^n t^{m-n} + b\hat{\beta}(s\beta)^n t^{m-n} + c\hat{\gamma}(s\gamma)^n t^{m-n})$$

Note that, for any real numbers a and b, and any positive integer m, the identity

$$(a+b)^m = \sum_{n=0}^{m}\binom{m}{n}a^n b^{m-n} \tag{14}$$

holds. Hence

$$a\hat{\alpha}(s\alpha+t)^m + b\hat{\beta}(s\beta+t)^m + c\hat{\gamma}(s\gamma+t)^m$$

$\alpha^3 = s\alpha + t$, $\beta^3 = s\beta + t$ and $\gamma^3 = s\gamma + t$ are due to (7). Hence,

$$a\hat{\alpha}\alpha^{3m} + b\hat{\beta}\beta^{3m} + c\hat{\gamma}\gamma^{3m}$$

Thus, the proof is completed. ■

Theorem 8. *Let m be a positive integer. Then,*

$$\sum_{n=0}^{m}\binom{m}{n}s^n t^{m-n}\mathcal{SR}_n = \mathcal{SR}_{3m}$$

Proof: Applying Binet-like formula (13) and combining this with (14) and (7), let $\sum_{n=0}^{m}\binom{m}{n}s^n t^{m-n}\mathcal{SR}_n = D$. We obtain the identity

$$D = \sum_{n=0}^{m}\binom{m}{n}s^n t^{m-n}(\hat{\alpha}\alpha^n + \hat{\beta}\beta^n + \hat{\gamma}\gamma^n)$$
$$= \sum_{n=0}^{m}\binom{m}{n}(\hat{\alpha}(s\alpha)^n t^{m-n} + \hat{\beta}(s\beta)^n t^{m-n} + \hat{\gamma}(s\gamma)^n t^{m-n})$$
$$= \hat{\alpha}(s\alpha+t)^m + \hat{\beta}(s\beta+t)^m + \hat{\gamma}(s\gamma+t)^m$$
$$= \hat{\alpha}\alpha^{3m} + \hat{\beta}\beta^{3m} + \hat{\gamma}\gamma^{3m}$$

Thus, the proof is completed. ■

Theorem 9. *Let m be a positive integer. Then,*

$$\sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^k\mathcal{SP}_{n-k} = \mathcal{SP}_{n+2m}$$

Proof: Applying Binet-like formula (12) and combining this with (14) and (7), let $\sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^k\mathcal{SP}_{n-k} = E$. We obtain the identity

$$E = \sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^k(a\hat{\alpha}\alpha^{n-k} + b\hat{\beta}\beta^{n-k} + c\hat{\gamma}\gamma^{n-k})$$
$$= \sum_{k=0}^{m}\binom{m}{k}(a\hat{\alpha}(s\alpha)^{m-k}t^k\alpha^{n-m} + b\hat{\beta}(s\beta)^{m-k}t^k\beta^{n-m}$$
$$+ c\hat{\gamma}(s\gamma)^{m-k}t^k\gamma^{n-m})$$
$$= a\hat{\alpha}(s\alpha+t)^m\alpha^{n-m} + b\hat{\beta}(s\beta+t)^m\beta^{n-m}$$
$$+ c\hat{\gamma}(s\gamma+t)^m\gamma^{n-m}$$
$$= a\hat{\alpha}\alpha^{n+2m} + b\hat{\beta}\beta^{n+2m} + c\hat{\gamma}\gamma^{n+2m}$$

Thus, the proof is completed. ■

Theorem 10. *Let m be a positive integer. Then,*

$$\sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^k\mathcal{SR}_{n-k} = \mathcal{SR}_{n+2m}$$

Proof: Applying Binet-like formula (13) and combining this with (14) and (7), let $\sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^k\mathcal{SR}_{n-k} = F$. we obtain the identity

$$F = \sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^k(\hat{\alpha}\alpha^{n-k} + \hat{\beta}\beta^{n-k} + \hat{\gamma}\gamma^{n-k})$$
$$= \sum_{k=0}^{m}\binom{m}{k}(\hat{\alpha}(s\alpha)^{m-k}t^k\alpha^{n-m} + \hat{\beta}(s\beta)^{m-k}t^k\beta^{n-m}$$
$$+ \hat{\gamma}(s\gamma)^{m-k}t^k\gamma^{n-m})$$
$$= \hat{\alpha}(s\alpha+t)^m\alpha^{n-m} + \hat{\beta}(s\beta+t)^m\beta^{n-m}$$
$$+ \hat{\gamma}(s\gamma+t)^m\gamma^{n-m}$$
$$= \hat{\alpha}\alpha^{n+2m} + \hat{\beta}\beta^{n+2m} + \hat{\gamma}\gamma^{n+2m}$$

Thus, the proof is completed. ■

III. Conclusion

The Fibonacci, Padovan and Perrin, and their generalizations play an important role in mathematics applied science, art and architectural, etc. We firstly take into consideration split quaternions and basic operations while preparing this work. Then, we define the new generalizations of the Padovan and Perrin as the split (s,t)-Padovan and (s,t)-Perrin quaternions. We give their Binet-like formulas and the generating functions. Also we obtain certain binomial sums and identities for them.

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