Numerical simulations used to detect the chaotic evolution of the exchange rate described by a nonlinear determinist system

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Abstract**—** In this paper we present a study concerning the exchange rate evolution governed by a third-order nonlinear determinist discrete system. We present some results concerning the unique fixed point of the system, its stability and its attraction domain under certain values of parameters and we also present the existence of period-two cycles. Given the nonlinear nature of the system, its most complex type of behavior is the chaotic dynamics. We cannot detect this type of behavior using only analytical tools. For this reason, in order to detect the dynamics of the system, we will use numerical simulations. It is known that the Lyapunov exponents are a tool used to establish the type of behavior in nonlinear dynamics. We will calculate their values, in order to establish the type of dynamics. From our numerical simulations, we present a case in which the system displays a chaotic behavior. For this particular case we also consider a corresponding system of the second order. From the images of the figures in this paper we can observe a similarity between the images of attractors for each particular order of the systems.

Keywords**—** nonlinear system, numerical simulations, chaos, attractors.

I. Introduction

According to [2] a general equation modeling the exchange rate evolution is given by:

$$S_t = X_t E_t(S_{t+1})^b \tag{1}$$

In the above equation, S_t is the exchange rate at the moment *t*; X_t describes the exogenous variables that drive the exchange rate at the moment *t*; $E_t(S_{t+1})$ is the expectation held at the moment *t* in the market about the exchange rate at the moment *t+1*; *b* is the discount factor that speculators use to discount the future expected exchange rate *(0<b<1).*

This model allows us to take into account two components for forecasting: a forecast made by the chartists $E_{ct}(S_{t+1})$ and a forecast made by the fundamentalists $E_{ft}(S_{t+1})$:

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$$E_t(S_{t+1})/S_{t-1} = \left(E_{ct}(S_{t+1})/S_{t-1}\right)^{m_t}\left(E_{ft}(S_{t+1})/S_{t-1}\right)^{1-m_t} \tag{2}$$

where m_t is the weight given by the chartists and $1-m_t$ is the weight given by the fundamentalists at the moment *t*.

The fundamentalists assume the existence of an equilibrium exchange rate S^*. If at the moment *t-1* the exchange rate S_{t-1} is above, respectively below the equilibrium rate S^*, the fundamentalists expect the future exchange rate S_{t+1} to go down, respectively increase with the speed α. More precisely, if they observe a deviation today, then their forecasts is the following:

$$\frac{E_{ft}(S_{t+1})}{S_{t-1}} = \left(\frac{S^*}{S_{t-1}}\right)^{\alpha}, \quad \alpha > 0 \tag{3}$$

The chartists use the past values of the exchange rate to detect patterns that they extrapolate in the future. An equation which gives a general description of the different models used by chartists is the following:

$$\frac{E_{ct}(S_{t+1})}{S_{t-1}} = f(S_{t-1},\ldots,S_{t-N}). \tag{4}$$

According to [2] it is possible to specify such a rule in general terms, as follows:

$$\frac{E_{ct}(S_{t+1})}{S_{t-1}} = \left(\frac{S_{t-1}}{S_{t-2}}\right)^{C_1}\left(\frac{S_{t-2}}{S_{t-3}}\right)^{C_2}\cdots\left(\frac{S_{t-N+1}}{S_{t-N}}\right)^{C_{N-1}} \tag{5}$$

The exact nature of this rule is determined by the coefficients c_i. These can be positive, negative, or zero.

The weight m_t, in equation (2), given by chartists is

$$m_t = \frac{1}{1+\beta\left(S_{t-1}-S^*\right)^2}, \quad \beta > 0 \tag{6}$$

The parameter β measures the precision degree of the fundamentalists' estimation. When the exchange rate is in the neighbourhood of the equilibrium rate, chartists' behavior dominates. When the exchange rate differs from the fundamental rate, then the expectation will be dominated by the fundamentalists.

In this paper we consider the case $X_t = 1$ (which means that $S^* = 1$) and for chartists we consider the expectation:

$$\frac{E_{ct}(S_{t+1})}{S_{t-1}} = \left(\frac{S_{t-1}}{S_{t-3}}\right)^2 \tag{7}$$

In equation (2) we will use the expectations given by the equations (3) and (7). In equation (1) we will use the expectations given by equation (2). In this way, we obtain the following difference equation:

$$S_t = S_{t-1}^{\left(\frac{(2+\alpha)b}{1+\beta(S_{t-1}-1)^2}+(1-\alpha)b\right)} S_{t-3}^{\left(\frac{-2b}{1+\beta(S_{t-1}-1)^2}\right)} \tag{8}$$

In *Section 2* we will present some analytical results for equation (8) and in *Section 3* we will present some numerical simulations.

II. THE EXCHANGE RATE EVOLUTION GOVERNED BY A THIRD-ORDER NONLINEAR DETERMINIST DISCRETE SYSTEM

If we denote $s_t = \ln S_t$, then equation (8) can be written in the form:

$$s_t = \left(\frac{(2+\alpha)b}{1+\beta\left(e^{s_{t-1}}-1\right)^2} + (1-\alpha)b\right)s_{t-1} + \left(\frac{-2b}{1+\beta\left(e^{s_{t-1}}-1\right)^2}\right)s_{t-3} \tag{9}$$

with $s_t \in R$ and $t \in Z$. We can rewrite equation (9) in the following vectorial form:

$$(s_t, s_{t+1}, s_{t+2}) = F(s_{t-1}, s_t, s_{t+1}) \tag{10}$$

where $F : R^3 \to R^3$, $F(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$, is defined in the following way: $F_1(x,y,z) = z$, $F_2(x,y,z) = y$ and $F_3(x,y,z) = \varphi(z)z + \psi(z)x$, with $\varphi(z) = \frac{(2+\alpha)b}{1+\beta(e^z-1)^2} + (1-\alpha)\beta$ and $\psi(z) = \frac{-2b}{1+\beta(e^z-1)^2}$.

A. *Steady-state existence, unicity and stability*

A fixed point for system (10) is a point (x^*, x^*, x^*) for which $(x^*, x^*, x^*) = F(x^*, x^*, x^*)$.

We recall that a fixed point (x^*, x^*, x^*) is stable if for any sufficiently small neighbourhood $U \ni (x^*, x^*, x^*)$ there is a neighbourhood $V_U(\alpha, b, \beta) \ni (x^*, x^*, x^*)$ such that $F^t(x', y', z') \in U$ for every point $(x', y', z') \in V_U(\alpha, b, \beta)$ and all *t > 0*, where $F^t = \underbrace{F \circ \ldots \circ F}_{t-times}$.

If there is a neighborhood $V_U(\alpha, b, \beta) \ni (x^*, x^*, x^*)$ so that $F^t(x', y', z') \to (x^*, x^*, x^*)$, when $t \to \infty$, for every point $(x', y', z') \in V_U(\alpha, b, \beta)$, then the fixed point is asymptotically stable (attracting fixed point).

Proposition 1 In the case in which $b \in (0,1)$, $\alpha > 0$ and $\beta > 0$, system (10) has a unique fixed point and this point is *(0,0,0)*. The fixed point *(0,0,0)* is stable for $b \in (0,\, 0.3165)$ and unstable for $b \in (0.3165,\, 1)$.

The equilibrium exchange rate means that money demand is equal to money supply. When the discount factor *b* is in the interval *(0, 0.3165)* the fixed point is stable. In fact, if *b* is small, this means that the exchange rate value is strongly influenced by the exogenous variables (which yield the equilibrium value of the exchange rate). We have seen that in this case it is important to know the equilibrium value, because the traders expect that in a neighborhood of equilibrium exchange rate, the exchange rates will go back to this value for $b \in (0,\, 0.3165)$.

For *b = 0.3165*, we notice a bifurcation. After the bifurcation, the fixed point is unstable and it is surrounded by a limit cycle that is stable (we observe this from simulations). Within a neighbourhood of the fixed point, all the orbits starting outside or inside the closed invariant curve, except at the origin, tend towards the limit cycle under the iterations of the function *F*. This is a Neimark-Sacker bifurcation.

Proposition 2 For $\alpha \in (0,1)$, $\beta > 0$ and $b \in (0,\, 0.2]$ the fixed point (0,0,0) of system (10) is globally attractive.

B. *Period-two cycles*

We shall now study the existence of cycles of period two.

Proposition 3 i) Under the assumptions $b \in (0,1)$, if $\alpha \in \left(0, 1+\frac{1}{b}\right]$ and $\beta > 0$ or if $\alpha \in \left(1+\frac{1}{b}, \infty\right)$ and $\beta \in \left(0, \frac{(\alpha-1)b^2+1}{(\alpha-1)^2b^2-1}\right]$, then the system (10) has no cycles of period two.

ii) If $b \in (0,1)$, $\alpha \in \left(1+\frac{1}{b}, \infty\right)$ and $\beta \in \left(\frac{(\alpha-1)b^2+1}{(\alpha-1)^2b^2-1}, \infty\right)$, then system (10) has only one cycle of period two. This cycle is $\{(s_1, s_2, s_1), (s_2, s_1, s_2)\}$ where s_1, s_2 are the solutions of the equation $\dfrac{\varphi\left(\dfrac{\varphi(x)}{1-\psi(x)}x\right)}{1-\psi\left(\dfrac{\varphi(x)}{1-\psi(x)}x\right)}\dfrac{\varphi(x)}{1-\psi(x)}x=1$, which means that

s_1, s_2 are the solutions of the equation

$$\frac{3b+(1-\alpha)b\beta\left(e^{\frac{3b+(1-\alpha)b\beta\left(e^x-1\right)^2}{1+2b+\beta\left(e^x-1\right)^2}}-1\right)^2}{1+2b+\beta\left(e^{\frac{3b+(1-\alpha)b\beta\left(e^x-1\right)^2}{1+2b+\beta\left(e^x-1\right)^2}}-1\right)^2}\frac{3b+(1-\alpha)b\beta\left(e^x-1\right)^2}{1+2b+\beta\left(e^x-1\right)^2}=1$$

The numbers s_1, s_2 verify the relation $s_1s_2<0$. Let s_1 be the positive number.

If $\beta\in\left(\dfrac{(\alpha-1)b^2+1}{(\alpha-1)^2b^2-1},\dfrac{1+b}{(\alpha-1)b-1}\right)$, then

$s_1>\ln\left(1+\sqrt{\dfrac{1}{\beta}\dfrac{1+b}{(\alpha-1)b-1}}\right)$ and $s_2<-\ln\left(1+\sqrt{\dfrac{1}{\beta}\dfrac{1+b}{(\alpha-1)b-1}}\right)$.

If $\beta\in\left(\dfrac{1+b}{(\alpha-1)b-1},\infty\right)$, then

$$s_1\in\left(\ln\left(1+\sqrt{\frac{1}{\beta}\frac{1+b}{(\alpha-1)b-1}}\right),-\ln\left(1-\sqrt{\frac{1}{\beta}\frac{1+b}{(\alpha-1)b-1}}\right)\right)$$

and

$$s_2\in\left(\ln\left(1-\sqrt{\frac{1}{\beta}\frac{1+b}{(\alpha-1)b-1}}\right),-\ln\left(1+\sqrt{\frac{1}{\beta}\frac{1+b}{(\alpha-1)b-1}}\right)\right).$$

From *Propositions 1* and *3*, we obtain the following proposition:

Proposition 4 If $b\in(0,\,0.3165)$, $\alpha\in\left(1+\dfrac{1}{b},\infty\right)$ and $\beta\in\left(\dfrac{(\alpha-1)b^2+1}{(\alpha-1)^2b^2-1},\infty\right)$, then the fixed point *(0,0,0)* of system (10) is only locally attractive.

III. NUMERICAL SIMULATIONS

A. Numerical simulations for the system (10)

We now recall some notions which will be used in this section. We say that a set *A* is an attracting set with the fundamental neighbourhood *U*, if it verifies the following properties (see [5]):

1) attractivity: for every open set $V\supset A$, $F^tU\subset V$ for all sufficiently large *t*.

2) invariance: $F^t(A)=A$, for all *t*.

3) *A* is minimal: there is no proper subset of *A* that satisfies conditions 1 and 2.

The basin of attraction is the set of initial points *x* so that $F^t(x)$ is close to *A* when $t\to\infty$.

It is possible to classify the different attractors: attracting fixed point, attracting n-cycle, quasiperiodic attractor and strange attractor. An attractor, as an experimental object, gives a global description of the asymptotic behavior of a dynamical system.

When a deterministic mechanism presents complex behavior with intermittence we can conclude that the series evinces chaos under certain conditions.

The sensitive dependence on initial conditions is one of the most essential aspects to identify chaos. We recall that the sensitive dependence on initial conditions means that two trajectories starting very close together will rapidly diverge from each other.

The strange attractor is associated with a chaotic state of time evolution and is characterized by the sensitive dependence on initial conditions.

A measure of the average rate of exponential divergence exhibited by a chaotic system is given by the Lyapunov exponents of the system; the positivity of one from these exponents can suggest the presence of chaos.

The Lyapunov exponents λ_1, λ_2 and λ_3 are given by

$$\left\{e^{\lambda_1},e^{\lambda_2},e^{\lambda_3}\right\}=\lim_{n\to\infty}\left\{eigenvalues\ of\left(\prod_{t=0}^{n-1}J\left(F\left(s_t,s_{t+1},s_{t+2}\right)\right)\right)^{\frac{1}{n}}\right\}\qquad(11)$$

where $J\left(F\left(s_t,s_{t+1},s_{t+2}\right)\right)$ represents the Jacobian matrix of the function *F*. For a period-*p* point the Lyapunov exponents λ_1, λ_2 and λ_3 are given by

$$\left\{e^{\lambda_1},e^{\lambda_2},e^{\lambda_{23}}\right\}=\left\{eigenvalues\ of\left(\prod_{t=0}^{p-1}J\left(F\left(s_t,s_{t+1},s_{t+2}\right)\right)\right)^{\frac{1}{p}}\right\}\qquad(12)$$

We recall now that for an attracting period-*p* cycle the Lyapunov exponents are negative; in case of a bifurcation point, at least one Lyapunov exponent is zero; for a limit cycle one Lyapunov exponent is zero and the others are negative and for a chaotic behavior the highest Lyapunov exponent is positive while the sum of the all Lyapunov exponents is negative.

In order to compute the Lyapunov exponents, when system (10) displays a chaotic behavior, we use the method proposed in [1], based on the Householder QR factorization and the implementation method proposed in [8].

We have made many numerical simulations and we have found many situations in which the system displays a chaotic behavior.

For the particular case where *α=2, b=0.95* and the initial condition $(s_0,s_1,s_2)=(0.02,-0.02,s_*)$, where $s_*=\left(\dfrac{(2+\alpha)b}{1+\beta\left(e^{s_1}-1\right)^2}+(1-\alpha)b\right)s_1+\left(\dfrac{-2b}{1+\beta\left(e^{s_1}-1\right)^2}\right)s_0$, we investigate the ranges of parameter β for which system (10) presents a chaotic or a non chaotic behavior.

We observe different intervals of values for β for which, in general, system (10) displays a chaotic behavior. These intervals are separated by an interval of values of β which characterizes a sequence of period-doubling bifurcations for system (10). In *Figures 1-3* we present the strange attractors which characterize different types of intervals of values for β for which the system displays a chaotic behavior.

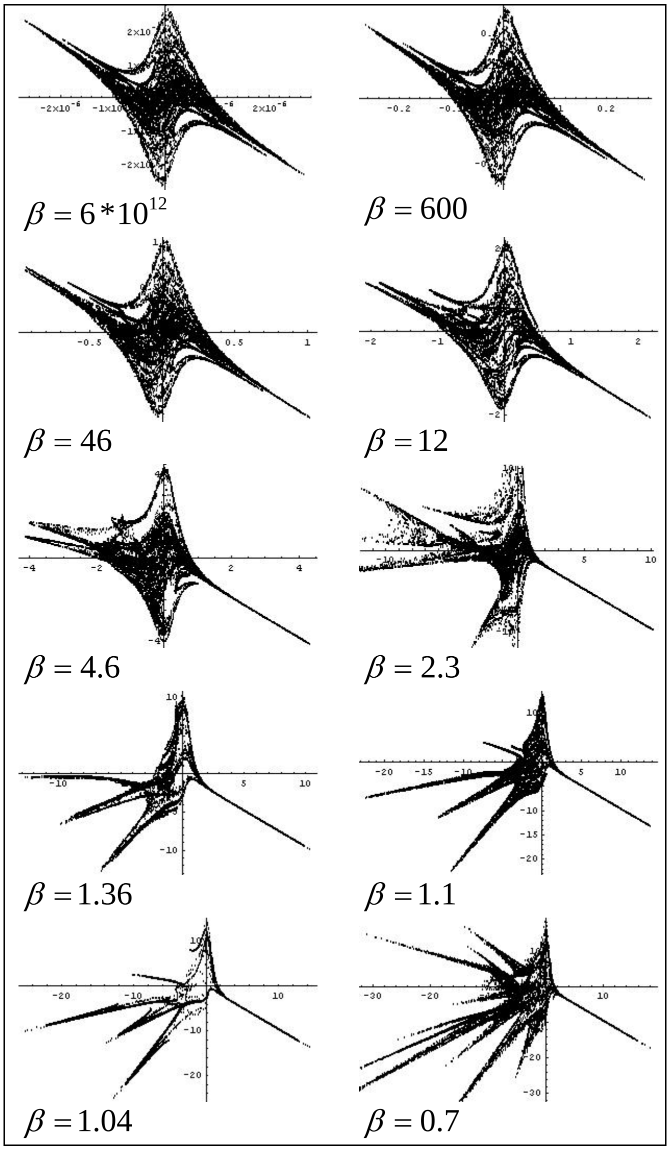

Fig. 1 Chaotic attractors in the case *b =0.95*, $\alpha = 2$, $(s_0, s_1, s_2) = (0.02, -0.02, s_*)$, in the space (s_t, s_{t+1})

In *Table 1* we give the values of Lyapunov exponents in the case of the strange attractors presented in *Figures 1-3*.

The images from these figures seem to represent the same attractor which increases and is deformed.

In this case, the fixed point *(0,0,0)* is unstable and system (10) has no cycles of period two. We observe more values for β for which system (10) displays a chaotic behavior.

When $b \in (0, 0.3165)$ (the fixed point *(0,0,0)* is unstable) the influence of speculators increases more and more. For certain values of the parameters α and β and of the exchange rate, the behavior is expected to be chaotic. This means that the influence of speculators increases and produces instability and the forecast of the exchange rate evolution is difficult.

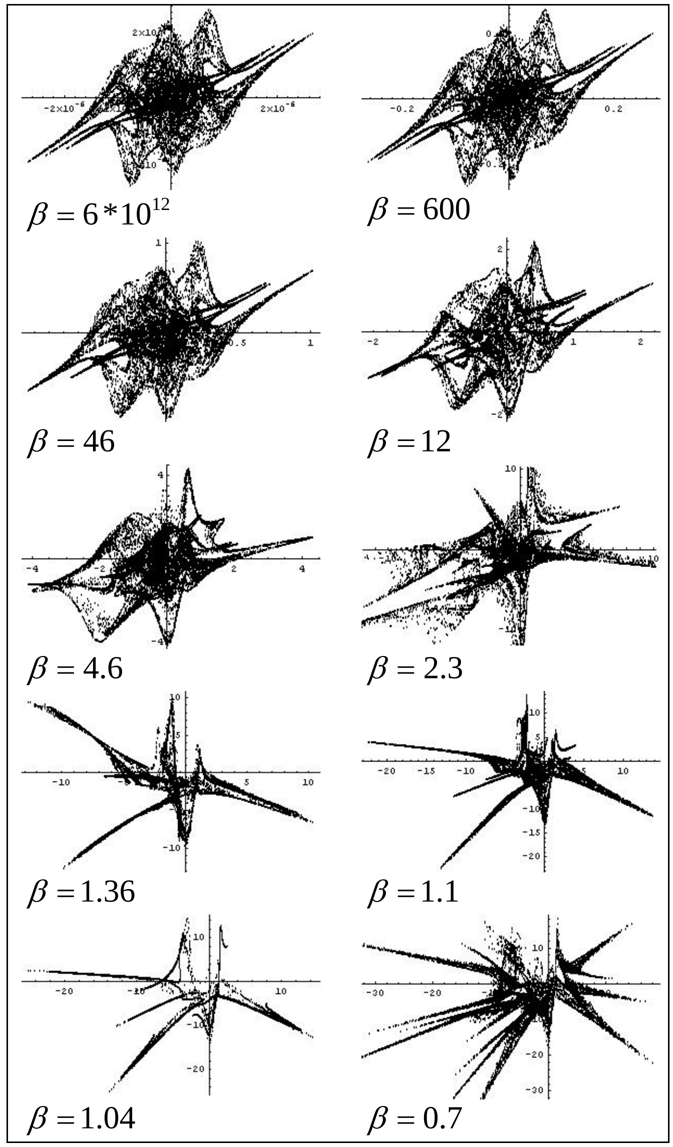

Fig. 2 Chaotic attractors in the case b =0.95, $\alpha = 2$, $(s_0, s_1, s_2) = (0.02, -0.02, s_*)$, in the space (s_t, s_{t+2})

Parameter β also influences the dynamics of the system. The speculators have a high influence on the market and create instability. We can see how fast the attractors tend towards the equilibrium value when β decreases.

When β is high the evolution of the exchange rate is around the equilibrium value and the weight given by the fundamentalists tends towards its maximum value 1. When β

decreases, this weight also decreases, and the values of the exchange rate and the value of equilibrium are not close.

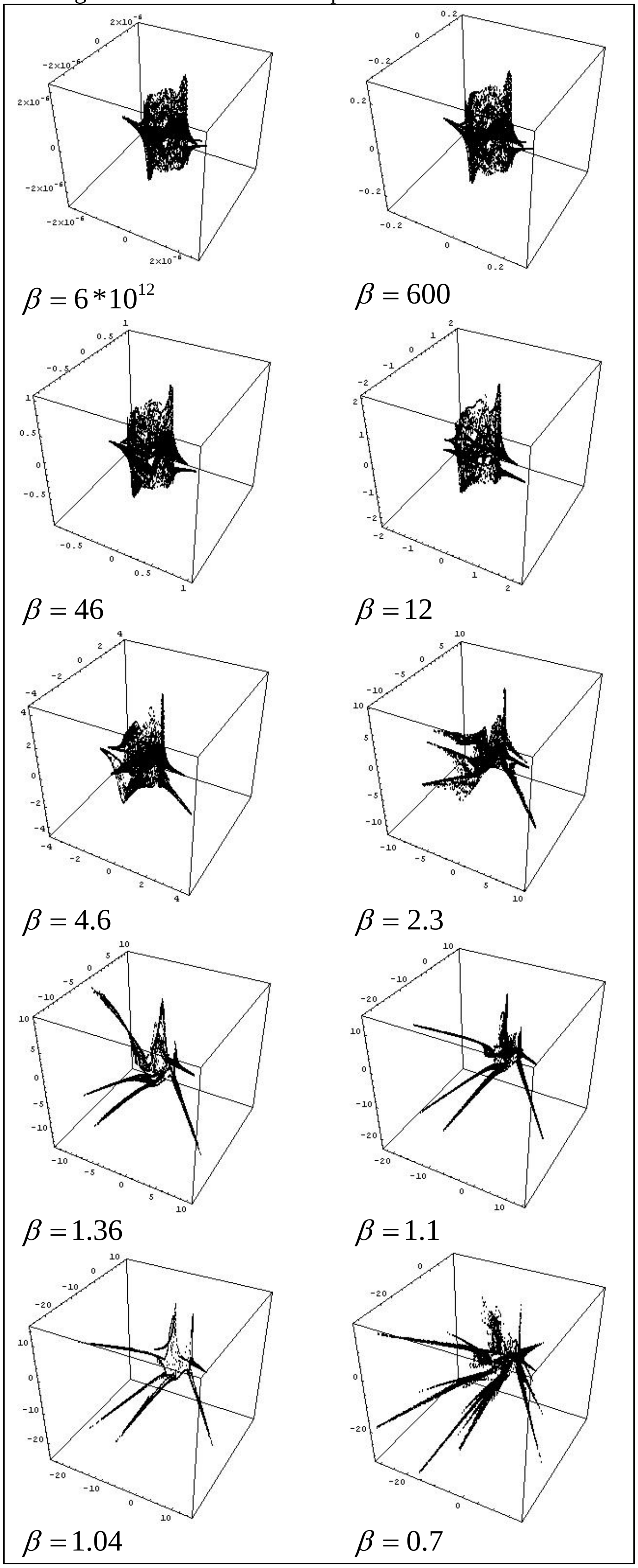

Fig. 3 Chaotic attractors in the case $b = 0.95$, $\alpha = 2$, $(s_0, s_1, s_2) = (0.02, -0.02, s_*)$, in the space (s_t, s_{t+1}, s_{t+2})

TABLE 1 The Lyapunov exponents in the case $b = 0.95$, $\alpha = 2$, $(s_0, s_1, s_2) = (0.02, -0.02, s_*)$

β	λ_1	λ_2	λ_3
$6*10^{12}$	0.2387	-0.2572	-0.4318
600	0.2328	-0.2468	-0.4297
46	0.2165	-0.2078	-0.4558
12	0.1925	-0.2225	-0.5905
6.2	0.174	-0.1869	-0.7605
4.6	0.2005	-0.1857	-0.8187
2.3	0.1283	-0.1239	-1.831
1.36	0.1685	-0.2739	-1.5168
1.1	0.1702	-0.2348	-1.5861
1.04	0.1446	-0.4764	-2.0404
0.7	0.1207	-0.1013	-2.1739

To calculate the Lyapunov exponents we have used the implementation method proposed in [8], using a *VBA* (Visual Basic for Applications) program in *Excel*, and the images from *Figures 1-3* are made using *Mathematica*.

B. *Numerical simulations for a system of the second order*

Now, for chartists we consider the expectation:

$$\frac{E_{ct}(S_{t+1})}{S_{t-1}} = \left(\frac{S_{t-1}}{S_{t-3}}\right)^2 \tag{13}$$

In equation (2) we will use the expectations given by the equations (3) and (13). In equation (1) we will use the expectations given by equation (2). In this way, we obtain the following difference equation:

$$S_t = S_{t-1}^{\left(\frac{(2+\alpha)b}{1+\beta(S_{t-1}-1)^2}+(1-\alpha)b\right)} S_{t-2}^{\left(\frac{-2b}{1+\beta(S_{t-1}-1)^2}\right)} \tag{14}$$

If we denote $s_t = \ln S_t$, then equation (14) can be written in the form:

$$s_t = \left(\frac{(2+\alpha)b}{1+\beta\left(e^{s_{t-1}}-1\right)^2} + (1-\alpha)b\right) s_{t-1} + \left(\frac{-2b}{1+\beta\left(e^{s_{t-1}}-1\right)^2}\right) s_{t-2} \tag{15}$$

with $s_t \in R$ and $t \in Z$. We can rewrite equation (15) in the following vectorial form:

$$(s_t, s_{t+1}) = F(s_{t-1}, s_t) \tag{16}$$

where

$F : R^2 \to R^2$, $F(x, y) = (F_1(x, y), F_2(x, y))$,

is defined in the following way: $F_1(x, y) = y$ and $F_2(x, y) = \varphi(y)y + \psi(y)x$, with $\varphi(y) = \dfrac{(2+\alpha)b}{1+\beta\left(e^y - 1\right)^2} + (1-\alpha)\beta$

and $\psi(y) = \dfrac{-2b}{1+\beta\left(e^y - 1\right)^2}$.

For the particular case where $\alpha=2$, $b=0.95$, $c=2$ and the initial condition $(s_0, s_1) = (0.02, -0.02)$, we investigate the ranges of parameter β for which system (16) presents a chaotic or a non chaotic behavior.

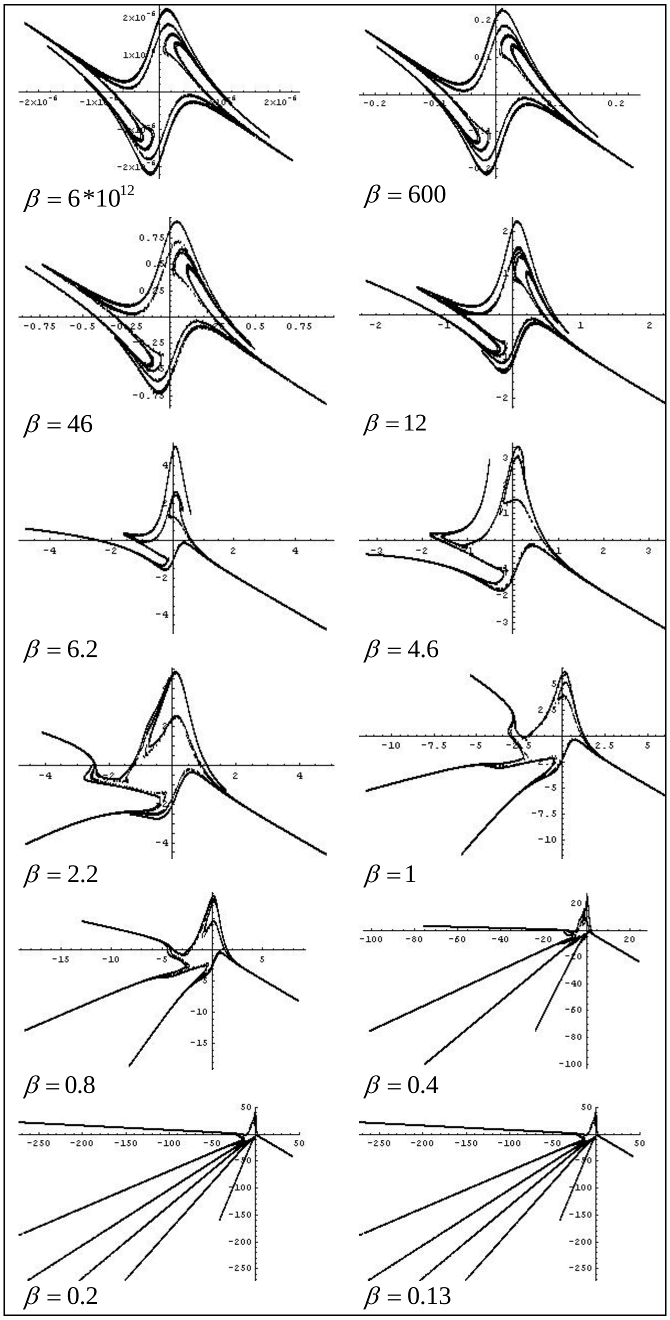

Fig. 4 Chaotic attractors in the case c=2 s_0 = *0.02*, s_1 = -0.02, *b =0.95*, α =2, space (s_t, s_{t+1})

We observe different intervals of values for β for which system (16), in general, displays a chaotic behavior. These intervals are [46, ∞), [16,28], [6.2,12], [4,4.6], [2.9,3.8], [1.7,1.8]. Every two intervals presented here are separated by an interval of values of β which characterizes a sequence of period-doubling bifurcations for system (16). We present in *Figure 4* the strange attractors which characterize different types of intervals of values for β for which the system displays a chaotic behavior. In *Table 2* we give the values of Lyapunov exponents in the case of the strange attractors present in *Figure 4*.

TABLE 2 Lyapunov exponents in the case s_0 *= 0.02,* s_1 *= -0.02, b =0.95,* α *=2*

β	λ_1	λ_2
$6*10^{12}$	0.3728	-1.2735
600	0.4029	-1.2158
46	0.2685	-1.1734
12	0.3299	-1.341
6.2	0.2636	-1.9012
4.6	0.2244	-1.4617
2.2	0.2033	-1.5614
1	0.09489	-1.424
0.8	0.143	-1.5599
0.4	0.1054	-3.3219
0.2	0.0787	-2.5733
0.13	0.0132	-4.1387

From *Figures 1* and *4* we can observe a similarity between the images of attractors for each particular order of the systems.

IV. CONCLUSION

Fixing the value of parameters and the initial condition, using numerical simulations, we can detect the dynamics displayed by the nonlinear systems.

From the case presented in the *Section 3*, we can observe that the dynamics of nonlinear systems can be very complicate. In such a study, the implementation methods are very important. If we use a good implementation method, we can quickly observe many cases (for example, fixing the parameters and initial condition and making only one parameter variable). In this way, we can conclude on the obtained results.

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