Integration method and Runge-Kutta method

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Abstract: In [1], M. Podisuk, U. Chundang and W. Sanprasert introduced the integration method for finding the numerical solution of the initial value problem of ordinary differential equation which turned out to be the multi-step method. In this paper, the single step method of this integration method with the help of Gauss-Legendre Quadrature integration formula and the Taylor series expansion is explored. The results of some examples that include the initial value problem of fourth order of ordinary differential equation are shown. These results are compared to the results that obtain by the Runge-Kutta method.

Keywords **Gauss-Quadrature-formula Gauss-Legendre-Quadrature-formula Integration-method Initial-value-problem Taylor series expansion Runge-Kutta-method**

1. Introduction

The initial value problem of ordinary differential equation is of the form $y'(x) = f(x,y)$, $x \in [a,b]$ (1)

with the initial condition $y(a) = y_0$. (2)

The above equations are equivalent to the integral equation

$$y(x) = y_0 + \int_a^x f(t,y)dt\,. \qquad (3)$$

Instead of solving the equations (1)-(2), we will solve the equation (3). Thus, we have the formula

$$y_{m+1} = y_m + \int_{x_m}^{x_{m+1}} f(x,y)dx\,. \qquad (4)$$

M. podisuk and W. Sanprasert, in [2,3], used the Newton-Cotes integration formula to find the numerical solution of equation (4). The Newton-Cotes integral formula of n points gives no error when the integrand is the polynomial of degree less than or equal to n-1. However the Gauss- Quadrature integral formula of n points gives no error when the integrand is the polynomial of degree less than or equal to 2n-1 which is better than the Newton-Cotes integral formula and the Gauss Quadrature integral formula is unequally spaced points formula so that this formula will not turn to be the multi-step formula.

2. Formulation

The n-point Gauss-Legendre quadrature integral formula is of the form

$$\int_{-1}^{1} f(x)dx \approx \sum_{k=1}^{n} A_k f(x_k) \qquad (5)$$

when x_k's are the roots of the Legendre orthogonal polynomial of degree $p_n(x)$, so we come up with the formula

$$y_{m+1} = y_m + \frac{1}{2}h\sum_{k=1}^{n} A_k f(x_m + hx_k, y(x_m + hx_k)) \qquad (6)$$

and we use Taylor's series expansion to approximate the value of $y(x_m + hx_k)$.

3. Example

The following notations are used for the various formulas.

The integration method;

I-1 for one point formula
I-2 for two points formula
I-3 for three points formula
I-4 for four points formula
I-5 for five points formula
I-6 for six points formula.

The Runge-Kutta method;

RK-2 for two points formula
RK-3 for three points formula(1-4-1)
RK-4 for four points formula(1-3-3-1)
RF-4 for Fehlberg formula(order 4)
RF-5 for Fehlberg formula(order 5).

Example 1. Find the solution of the equation $y'(x) = xe^{-x^2} - 2xy, x \in [0,1]$ with the initial condition $y(0) = 1$.
The analytical solution of the above equations is $y(x) = e^{-x^2}\left(1 + \frac{1}{2}x^2\right)$.

The following tables are the results of the indicated formulas.

$x = 0.1,\ h = 0.1,\ y = 0.9950000829$

Form.	Cal. value	Error
I-1	0.9950000156	6.730534×10^{-8}
I-2	0.9950000762	6.686605×10^{-9}
I-3	0.9950000831	2.100933×10^{-10}
I-4	0.9950000831	2.073648×10^{-10}
I-5	0.9950000831	2.073648×10^{-10}
I-6	0.9950000831	2.091838×10^{-10}
RK-2	0.9949502492	4.983375×10^{-5}
RK-3	0.9950085100	8.427110×10^{-6}
RK-4	0.9950028776	2.794671×10^{-6}
RF-4	0.9950000822	7.003109×10^{-10}
RF-5	0.9950000817	1.234184×10^{-10}

Table 1

$x = 0.1,\ h = 0.01,\ y = 0.9950000829$

Form.	Cal. value	Error
I-1	0.995000819	1.0240910×10^{-9}
I-2	0.9950000762	6.6866050×10^{-9}
I-3	0.9950000829	$1.8189894 \times 10^{-12}$
I-4	0.9950000829	$1.8189894 \times 10^{-12}$
I-5	0.9950000829	$9.0949470 \times 10^{-13}$
I-6	0.9950000829	$9.0949470 \times 10^{-12}$
RK-2	0.9949998112	2.7169244×10^{-7}
RK-3	0.9950000913	8.3427949×10^{-9}
RK-4	0.9950000857	2.7803253×10^{-9}
RF-4	0.9950000829	$9.0949470 \times 10^{-13}$
RF-5	0.9950000829	$9.0949470 \times 10^{-13}$

Table 2

$x = 0.1,\ h = 0.001,\ y = 0.9950000829$

Form.	Cal. value	Error
I-1	0.9950000819	1.0240910×10^{-9}
I-2	0.9950000829	2.7284841×10^{-9}
I-3	0.9950000829	$2.7284841 \times 10^{-12}$
I-4	0.9950000829	$2.7284841 \times 10^{-12}$
I-5	0.9950000829	$8.1854523 \times 10^{-12}$
I-6	0.9950000829	$3.6379788 \times 10^{-12}$
RK-2	0.9950000804	2.4892870×10^{-9}
RK-3	0.9950000829	$1.3642421 \times 10^{-11}$
RK-4	0.9950000829	$6.3664629 \times 10^{-12}$
RF-4	0.9950000829	$9.0949470 \times 10^{-13}$
RF-5	0.9950000829	$9.0949470 \times 10^{-13}$

Table 3

$x = 0.1,\ h = 0.0001,\ y = 0.9950000829$

Form.	Cal. value	Error
I-1	0.9950000829	$5.4569682 \times 10^{-12}$
I-2	0.9950000829	$2.7284841 \times 10^{-12}$
I-3	0.9950000829	$5.4569682 \times 10^{-12}$
I-4	0.9950000829	$5.4569682 \times 10^{-12}$
I-5	0.9950000829	$8.1854523 \times 10^{-12}$
I-6	0.9950000829	$3.6379788 \times 10^{-12}$
RK-2	0.9950000804	2.4892870×10^{-9}
RK-3	0.9950000829	$1.3642421 \times 10^{-11}$
RK-4	0.9950000829	$5.4569682 \times 10^{-12}$
RF-4	0.9950000829	$6.3664629 \times 10^{-12}$

RF-5	0.9950000829	5.4569682×10^{-12}

Table 4

$x = 1.0$, $h = 0.0001$ $y = 0.55181916068$

Form.	Cal. value	Error
I-1	0.5518191609	2.5647751×10^{-10}
I-2	0.5518191612	5.4569682×10^{-10}
I-3	0.5518191612	5.4569682×10^{-10}
I-4	0.5518191612	5.4569682×10^{-10}
I-5	0.5518191612	5.8025762×10^{-10}
I-6	0.5518191613	6.7666405×10^{-10}
RK-2	0.5518191610	3.3833202×10^{-10}
RK-3	0.5518191612	5.4569682×10^{-10}
RK-4	0.5518191612	5.4569682×10^{-10}
RF-4	0.5518191612	5.4569682×10^{-10}
RF-5	0.5518191612	5.4569682×10^{-10}

Table 5

Example 2. Find the solution of the equation $y'(x) = e^{y}$, $x \in [0,2]$ with the initial condition $y(0) = -\ln 3$.

The analytical solution of the above equations is $y(x) = \ln\left(\frac{1}{3-x}\right)$.

The following tables are the results of the indicated formulas.

$x = 0.1$, $h = 0.1$, $y = -1.064710737$

Form.	Cal. value	Error
I-1	-1.064714037	3.2995667×10^{-6}
I-2	-1.064710845	1.0796794×10^{-7}
I-3	-1.064710845	1.0786789×10^{-7}
I-4	-1.064710845	1.0786789×10^{-7}
I-5	-1.064710845	1.0786789×10^{-7}
I-6	-1.064710845	1.0786789×10^{-7}
RK-2	-1.064714037	3.2997850×10^{-6}
RK-3	-1.064710736	5.8207661×10^{-9}
RK-4	-1.064710737	2.7803253×10^{-11}
RF-4	-1.064710737	5.4569682×10^{-12}
RF-5	-1.064710737	0.0

Table 6

$x = 0.1$, $h = 0.01$, $y = -1.064710737$

Form.	Cal. value	Error
I-1	-1.064714737	1.6263584×10^{-8}
I-2	-1.064710737	1.0958354×10^{-10}
I-3	-1.064710737	1.1095835×10^{-10}
I-4	-1.064710737	1.1095835×10^{-10}
I-5	-1.064710737	1.0913936×10^{-10}
I-6	-1.064710737	1.1823431×10^{-10}
RK-2	-1.064710770	3.3032848×10^{-8}
RK-3	-1.064710737	0.0
RK-4	-1.064710737	0.0
RF-4	-1.064710737	0.0
RF-5	-1.064710737	0.0

Table 7

$x = 0.1$, $h = 0.001$, $y = -1.064710737$

Form.	Cal. value	Error
I-1	-1.064714737	1.6552804×10^{-10}
I-2	-1.064710737	1.1095835×10^{-10}
I-3	-1.064710737	5.4569682×10^{-12}
I-4	-1.064710737	5.4569682×10^{-12}
I-5	-1.064710737	1.0913936×10^{-10}
I-6	-1.064710737	1.1823431×10^{-10}
RK-2	-1.064710770	3.3032848×10^{-8}
RK-3	-1.064710737	5.4569682×10^{-12}
RK-4	-1.064710737	0.0
RF-4	-1.064710737	5.4569682×10^{-12}
RF-5	-1.064710737	0.0

Table 8

$x = 0.1$, $h = 0.0001$, $y = -1.064710737$

Form.	Cal. value	Error
I-1	-1.064710737	1.2732926×10^{-11}
I-2	-1.064710737	1.4551915×10^{-11}
I-3	-1.064710737	5.4569682×10^{-12}
I-4	-1.064710737	5.4569682×10^{-12}
I-5	-1.064710737	8.7311491×10^{-11}
I-6	-1.064710737	2.0008883×10^{-11}
RK-2	-1.064710737	1.8189894×10^{-11}
RK-3	-1.064710737	1.4551915×10^{-11}
RK-4	-1.064710737	1.4551915×10^{-11}
RF-4	-1.064710737	1.4551915×10^{-11}

RF-5	-1.064710737	1.4551915×10^{-11}

Table 9

$x = 2.0$, $h = 0.0001$, $y = 0.0$

Form.	Cal. value	Error
I-1	4.7904818×10^{-9}	4.79048178×10^{-9}
I-2	-2.632585×10^{-10}	2.632585×10^{-10}
I-3	-2.632718×10^{-10}	2.632718×10^{-10}
I-4	-2.630020×10^{-10}	2.630020×10^{-10}
I-5	4.483292×10^{-9}	4.483292×10^{-9}
I-6	4.656209×10^{-9}	4.656209×10^{-9}
RK-2	-8.155041×10^{-10}	8.155041×10^{-10}
RK-3	2.628106×10^{-10}	2.628106×10^{-10}
RK-4	-2.637236×10^{-10}	2.637236×10^{-10}
RF-4	-2.593646×10^{-10}	2.593646×10^{-10}
RF-5	-2.578687×10^{-10}	2.578687×10^{-10}

Table 10

In the first two examples, we approximate the value of $y(x_m + hs)$ by

$$y(x_m + hs) = y(x_m) + hsf(x_m, y(x_m)) + \frac{1}{2}h^2s^2\begin{bmatrix} f_x(x_m, y(x_m)) \\ + f(x_m, y(x_m))f_y(x_m, y(x_m)) \end{bmatrix}.$$

Example3. Find the solution of the equation

$$y^{iv} = \frac{y + y' + y'' + y''' + 10e^x}{4} \quad x \in [0,2]$$

with the initial conditions

$y(1) = 0, y'(0) = 1, y''(0) = 2, y'''(0) = 3$.

The analytical solution of the above equations is $y(x) = xe^x$.

Let

$$y_1 = y, \qquad y_2 = y' = y_1',$$
$$y_3 = y'' = y_2', \qquad y_4 = y''' = y_3'$$
$$y_4' = y^{(iv)} = f(x, y_1, y_2, y_3, y_4)$$
$$= \frac{y_1 + y_2 + y_3 + y_4 + 10e^x}{4}$$

$$y^{(v)} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y_1}\frac{\partial y_1}{\partial x} + \frac{\partial f}{\partial y_2}\frac{\partial y_2}{\partial x} + \frac{\partial f}{\partial y_3}\frac{\partial y_3}{\partial x} + \frac{\partial f}{\partial y_4}\frac{\partial y_4}{\partial x}$$

$$y^{(v)} = \frac{50e^x}{16} + \frac{y_1}{16} + \frac{5y_2}{16} + \frac{5y_3}{16} + \frac{5y_4}{16}$$

$$y^{(vi)} = \frac{250e^x}{64} + \frac{5y_1}{64} + \frac{9y_2}{64} + \frac{25y_3}{64} + \frac{25y_4}{64}$$

$$y^{(vii)} = \frac{500e^x}{256} + \frac{25y_1}{256} + \frac{45y_2}{256} + \frac{61y_3}{256} + \frac{125y_4}{256}.$$

We approximate $y_1(x_m + p)$, $y_2(x_m + q)$, $y_3(x_m + r)$, and $y_4(x_m + s)$ by

$$y_1(x_m + hp) = y_1(x_m) + hpy_1'(x_m) + \frac{1}{2}h^2p^2y_1''(x_m) + \frac{1}{6}h^3p^6y_1'''(x_m) + \frac{1}{24}h^4p^4y_1^{(iv)}(x_m)$$

$$y_2(x_m + hq) = y_2(x_m) + hqy_2'(x_m) + \frac{1}{2}h^2q^2y_2''(x_m) + \frac{1}{6}h^3q^6y_2'''(x_m) + \frac{1}{24}h^4q^4y_2^{(iv)}(x_m)$$

$$y_3(x_m + hr) = y_3(x_m) + hry_3'(x_m) + \frac{1}{2}h^2r^2y_3''(x_m) + \frac{1}{6}h^3r^6y_3'''(x_m) + \frac{1}{24}h^4r^4y_3^{(iv)}(x_m)$$

$$y_4(x_m + sr) = y_4(x_m) + hsy_4'(x_m) + \frac{1}{2}h^2s^2y_4''(x_m) + \frac{1}{6}h^3s^6y_4'''(x_m) + \frac{1}{24}h^4s^4y_4^{(iv)}(x_m).$$

The following tables are the results of the indicated formulas.

$x = 0.1$, $h = 0.1$, $y = 0.1105170918$

Form.	Cal. value	Error
I-1	0.110383464	1.33628266×10^{-4}
I-2	0.110517072	2.00481054×10^{-8}
I-3	0.110517083	8.47421688×10^{-9}
I-4	0.110517083	8.47410397×10^{-9}
I-5	0.110517083	8.47410397×10^{-9}
I-6	0.110517083	8.47410397×10^{-9}
RK-2	0.110000000	5.17091808×10^{-4}
RK-3	0.110500000	1.70918074×10^{-5}
RK-4	0.110511111	5.98069630×10^{-6}
RF-4	0.110517150	5.86586566×10^{-8}
RF-5	0.110517088	3.75257514×10^{-9}

Table 11

$x = 0.1$, $h = 0.01$, $y = 0.1105170918$

Form.	Cal. value	Error
I-1	0.110515674	1.41810347×10^{-6}
I-2	0.110517092	1.36424205×10^{-12}
I-3	0.110517092	0.0
I-4	0.110517092	0.0
I-5	0.110517092	1.13686838×10^{-13}
I-6	0.110517092	0.0
RK-2	0.110511438	5.65346886×10^{-6}
RK-3	0.110517073	1.86772695×10^{-8}
RK-4	0.110517300	2.08467213×10^{-7}
RF-4	0.110517092	7.16227078×10^{-12}
RF-5	0.110517092	2.27373675×10^{-13}

Table 12

$x = 0.1$, $h = 0.001$, $y = 0.1105170918$

Form.	Cal. value	Error
I-1	0.110515674	1.41810347×10^{-6}
I-2	0.110517092	2.04636308×10^{-12}
I-3	0.110517092	0.0
I-4	0.110517092	0.0
I-5	0.110517092	1.13686838×10^{-13}
I-6	0.110517092	0.0
RK-2	0.110511438	5.70385055×10^{-8}
RK-3	0.110517073	1.86772695×10^{-8}
RK-4	0.110517354	2.61695959×10^{-7}
RF-4	0.110517092	2.06910045×10^{-11}
RF-5	0.110517092	2.16004992×10^{-12}

Table 13

$x = 0.1$, $h = 0.0001$, $y = 0.1105170918$

Form.	Cal. value	Error
I-1	0.110517092	1.23463906×10^{-10}
I-2	0.110517092	2.06910045×10^{-11}
I-3	0.110517092	2.06910045×10^{-11}
I-4	0.110517092	2.06910045×10^{-11}
I-5	0.110517092	2.06910045×10^{-11}
I-6	0.110517092	2.08046913×10^{-11}
RK-2	0.110517091	5.48538992×10^{-10}
RK-3	0.110517092	2.06910045×10^{-11}
RK-4	0.110517358	2.66695959×10^{-7}
RF-4	0.110517092	2.16004992×10^{-12}
RF-5	0.110517092	2.16004992×10^{-12}

Table 14

$x = 2.0$, $h = 0.0001$, $y = 14.778112074$

Form.	Cal. value	Error
I-1	14.77811217	9.52131813×10^{-8}
I-2	14.77811220	1.24709914×10^{-7}
I-3	14.77811220	1.24739017×10^{-7}
I-4	14.77811220	1.24986400×10^{-7}
I-5	14.77811220	1.24811777×10^{-7}
I-6	14.77811220	1.24462501×10^{-7}
RK-2	14.77811220	1.22963684×10^{-8}
RK-3	14.77811220	1.86772695×10^{-7}
RK-4	14.85454040	7.64283215×10^{-2}
RF-4	14.77811220	1.24782673×10^{-7}
RF-5	14.77811220	1.24695362×10^{-7}

Table 15

4. Conclusion

The new integration method works well for both first order initial value problem of the ordinary differential equation and fourth order of initial value of ordinary differential equation. As we may see from the above numerical results, this integration method work as good as Runge-Kutta method. We strongly recommend the one point formula, the two points formula and the three points formula.

References

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