# Smoothing curve by orthogonal polynomials

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***Abstract:*** **The interpolation by polynomial of degree n, $p_n(x)$, may smooth any curve $y = f(x)$ with the condition that this polynomial must passes through $n+1$ points of the curve $y = f(x)$. This condition may make the shape of the curve of $y = p_n(x)$ different from the exact shape of curve of $y = f(x)$ especially in the case that the points of $f(x)$ are obtained from an experiment which may not be the exact point of the curve $y = f(x)$. But the least square approximation by orthogonal polynomial, $q_n(x)$, may give the better shape than the interpolation by polynomial, $p_n(x)$. However the curve of $y = q_n(x)$ may not pass through any exact point of the curve $y = f(x)$. M.Podisuk, P.Rattanathanawan and P.Phataranavik in [1] introduced the sequences of orthogonal polynomials with step functions as their weight functions but they did not use them for least square approximation. In this paper, the sequences of orthogonal polynomials with step functions as their weight functions will be used for least square approximation.**

***Keywords:*** **Interpolation Orthogonal-polynomial Least-square-approximation , Step-function Non-differentiable-function Smoothing-curve**

## 1. Introduction

The interpolation of the function, $f(x)$, by polynomial, $p_n(x)$, is of the form

$$p_n(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$$
$$= \sum_{k=1}^{n+1} L_k(x)f(x_k) \qquad (1)$$

where $L_k(x) = \dfrac{\prod_{\substack{j=1 \\ j\neq k}}^{n+1}(x - x_j)}{\prod_{\substack{j=1 \\ j\neq k}}^{n+1}(x_k - x_j)}$

or $p_n(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$

$$= f(x_1) + f[x_1, x_2](x - x1) + ...$$
$$+ f[x_1,..., x_{n+1}](x - x_1)...(x - x_n). \qquad (2)$$

The equation (1) is called Lagrange method and the equation (2) is called Newton divided difference method.

The least square approximation method of the function $f(x)$ by polynomial $q_n(x)$ is of the form

$$q_n(x) = d_0p_0(x) + ... + d_np_n(x) \qquad (3)$$

where $p_0(x),..., p_n(x)$ are the orthogonal polynomials with respect to the weight function $w(x)$ in the interval $[a, b]$ and

$$d_k = \frac{\int_a^b w(x)p_k(x)f(x)dx}{\int_a^b w(x)p_k(x)p_k(x)dx}. \quad (4)$$

## 2. Orthogonal Polynomials

In this paper, 6 sequences of orthogonal polynomials are constructed. The orthogonal polynomials up to degree 7 are obtained for each sequence of orthogonal polynomials.

The 6 weight functions are step functions and they are, respectively.

First weight function

$$w(x) = \begin{cases} \frac{2}{3} & ,0 \le x < \frac{1}{2} \\ \frac{4}{3} & ,\frac{1}{2} \le x \le 1 \end{cases}.$$

Second weight function

$$w(x) = \begin{cases} \frac{2}{3} & ,0 \le x < \frac{1}{3} \\ \frac{5}{3} & ,\frac{1}{3} \le x < \frac{2}{3} \\ \frac{2}{3} & ,\frac{2}{3} \le x \le 1 \end{cases}.$$

Third weight function

$$w(x) = \begin{cases} \frac{4}{3} & ,0 \le x < \frac{1}{3} \\ \frac{1}{3} & ,\frac{1}{3} \le x < \frac{2}{3} \\ \frac{4}{3} & ,\frac{2}{3} \le x \le 1 \end{cases}.$$

Fourth weight function

$$w(x) = \begin{cases} \frac{1}{2} & ,0 \le x < \frac{1}{3} \\ 1 & ,\frac{1}{3} \le x < \frac{2}{3} \\ \frac{3}{2} & ,\frac{2}{3} \le x \le 1 \end{cases}.$$

Fifth weight function

$$w(x) = \begin{cases} \frac{4}{3} & ,0 \le x < \frac{1}{2} \\ \frac{2}{3} & ,\frac{1}{2} \le x \le 1 \end{cases}.$$

Sixth weight function

$$w(x) = \begin{cases} \frac{3}{2} & ,0 \le x < \frac{1}{3} \\ 1 & ,\frac{1}{3} \le x < \frac{2}{3} \\ \frac{1}{2} & ,\frac{2}{3} \le x \le 1 \end{cases}.$$

The 6 sequences of orthogonal polynomials of the above weight functions are as follow (up to orthogonal polynomial of degree 7 for each sequence).

First Sequence.

$p_0(x) = 1 \qquad p_1(x) = x - 0.583333$

$p_2(x) = x^2 - 1.045455x + 0.193182$
$= (x - 0.239775)(x - 0.805680)$

$p_3(x) = x^3 - 1.562865x^2 + 0.663450x$
$- 0.058735$
$= (x - 0.119695)(x - 0.548453)$
$(x - 0.894718)$

$p_4(x) = x^4 - 2.050478x^3 + 1.364187x^2$
$- 0.318118x + 0.016791$
$= (x - 0.073460)(x - 0.356348)$
$(x - 0.687005)(x - 0.933665)$

$p_5(x) = x^5 - 2.55926x^4 + 2.341592x^3$
$- 0.909289x^2 + 0.134961x - 0.004676$
$= (x - 0.048754)(x - 0.240697)$
$(x - 0.535090)(x - 0.779689)$
$(x - 0.955031)$

$p_6(x) = x^6 - 3.052228x^5 + 3.542064x^4$
$- 1.938208x^3 + 0.49978x^2 - 0.051750x$
$+ 0.001278$
$= (x - 0.035044)(x - 0.176361)$
$(x - 0.403078)(x - 0.634239)$
$(x - 0.836211)(x - 0.967296)$

$p_7(x) = x^7 - 3.557836x^6 + 5.020652x^5$
$- 3.563667x^4 + 1.328383x^3$
$- 0.245662x^2 + 0.01872x - 0.000344.$
$= (x - 0.026180)(x - 0.133124)$
$(x - 0.307114)(x - 0.527847)$
$(x - 0.713416)(x - 0.874833)$
$(x - 0.975321).$

Second Sequence.

$p_0(x) = 1 \qquad p_1(x) = x - 0.5$
$p_2(x) = x^2 - x + 0.191358$
$= (x - 0.257839)(x - 0.742161)$
$p_3(x) = x^3 - 1.5x^2 + 0.607018x$
$- 0.053509$
$= (x - 0.121870(x - 0.5)(x - 0.878130)$
$p_4(x) = x^4 - 2x^3 + 1.298334x^2$
$- 0.298334x + 0.015525$
$= (x - 0.072397)(x - 0.362811)$
$(x - 0.637189)(x - 0.927603)$
$p_5(x) = x^5 - 2.5x^4 + 2.236575x^3$
$- 0.854863x^2 + 0.127203x - 0.004458$
$= (x - 0.049488)(x - 0.254107)$
$(x - 0.5)(x - 0.745893)(x - 0.950512)$
$p_6(x) = x^6 - 3x^5 + 3.417847x^4$
$- 1.835693x^3 + 0.465402x^2$
$- 0.047556x + 0.001162$
$= (x - 0.034710)(x - 0.175528)$
$(x - 0.398135)(x - 0.601866)$
$(x - 0.824472)(x - 0.965290)$
$p_7(x) = x^7 - 3.5x^6 + 4.860303x^5$
$- 3.400758x^4 + 1.254548x^3$
$- 0.231064x^2 + 0.017620x - 0.000324$
$= (x - 0.026249)(x - 0.134096)$
$(x - 0.322882)(x - 0.5)(x - 0.677118)$
$(x - 0.865904)(x - 0.973751).$

Third Sequence.

$p_0(x) = 1 \qquad p_1(x) = x - 0.5$
$p_2(x) = x^2 - x + 0.141975$
$= (x - 0.171329)(x - 0.828671)$
$p_3(x) = x^3 - 1.5x^2 + 0.596190x$
$- 0.048095$
$(x - 0.107814)(x - 0.5)(x - 0.892186)$
$p_4(x) = x^4 - 2x^3 + 1.261354x^2$
$- 0.261354x + 0.012003$
$= (x - 0.072397)(x - 0.362811)$
$(x - 0.634779)(x - 0.936522)$
$p_5(x) = x^5 - 2.5x^4 + 2.211451x^3$
$- 0.8171765545x^2 + 0.112928x$
$- 0.003601$
$= (x - 0.044673(x - 0.214910)$
$(x - 0.5)(x - 0.785002)(x - 0.955327)$
$p_6(x) = x^6 - 3x^5 + 3.390072x^4$
$- 1.7801444071x^3 + 0.431483x^2$
$- 0.041410x + 0.000944$
$= (x - 0.032307)(x - 0.160529)$
$(x - 0.338909)(x - 0.661091)$
$(x - 0.839471)(x - 0.967693)$
$p_7(x) = x^7 - 3.5x^6 + 4.828945x^5$
$- 3.322361x^4 + 1.186374x^3$
$- 0.207200x^2 + 0.014746x - 0.000252$
$= (x - 0.024290)(x - 0.122405)$
$(x - 0.271576)(x - 0.5)(x - 0.728244)$
$(x - 0.877592)(x - 0.975710).$

Fourth Sequence.

$p_0(x) = 1 \qquad p_1(x) = x - 0.61^{\bullet}$
$p_2(x) = x^2 - 1.086957x + 0.219807$
$= (x - 0.268594)(x - 0.818363)$
$p_3(x) = x^3 - 1.575064x^2 + 0.67966x$
$- 0.064084$
$(x - 0.130469)(x - 0.547572)$
$(x - 0.897024)$
$p_4(x) = x^4 - 2.090974x^3$
$+ 1.427080x^2 - 0.343890x + 0.018737$

$$= (x-0.075749)(x-0.375536)$$
$$(x-0.703746)(x-0.935942)$$
$$p_5(x) = x^5 - 2.577745x^4 + 2.384440x^3$$
$$-0.943315x^2 + 0.145037x - 0.005246$$
$$= (x-0.051048)(x-0.260651)$$
$$(x-0.525840)(x-0.784437)$$
$$(x-0.955769)$$
$$p_6(x) = x^6 - 3.083635x^5$$
$$+3.622265x^4 - 2.011090x^3 +$$
$$0.527733x^2 - 0.055842x + 0.001413$$
$$= (x-0.035939)(x-0.181868)$$
$$(x-0.410786)(x-0.647333)$$
$$(x-0.839772)(x-0.967943)$$
$$p_7(x) = x^7 - 3.584598x^6 + 5.105955x^5$$
$$-3.668220x^4 + 1.38953x^3 - 0.262583x^2$$
$$+0.020526x - 0.000387$$
$$= (x-0.026890)(x-0.137279)$$
$$(x-0.327550)(x-0.518986)$$
$$(x-0.721228)(x-0.876965)$$
$$(x-0.975701).$$

Fifth Sequence.

$$p_0(x) = 1 \qquad p_1(x) = x - 0.416^{\bullet}$$
$$p_2(x) = x^2 - 0.95^{\bullet}4^{\bullet}x + 0.1477^{\bullet}2^{\bullet}$$
$$= (x-0.194320)(x-0.760225)$$
$$p_3(x) = x^3 - 1.437135x^2 + 0.537719x$$
$$-0.041849$$
$$= (x-0.105282)(x-0.451547)$$
$$(x-0.880305)$$
$$p_4(x) = x^4 - 1.949522x^3$$
$$+1.212753x^2 - 0.258822x + 0.012382$$
$$= (x-0.066335)(x-0.312995)$$
$$(x-0.643652)(x-0.926540)$$
$$p_5(x) = x^5 - 2.440739x^4 + 2.104548x^3$$
$$-0.755922x^2 + 0.104115x - 0.003226$$
$$= (x-0.044969)(x-0.220311)$$
$$(x-0.464910)(x-0.759303)$$
$$(x-0.951246)$$
$$p_6(x) = x^6 - 2.947771x^5$$
$$+3.280920x^4 - 1.707760x^3$$
$$+0.415246x^2 - 0.040286x + 0.000929$$
$$= (x-0.032704)(x-0.163790)$$
$$(x-0.365761)(x-0.596922)$$
$$(x-0.823639)(x-0.964956)$$
$$p_7(x) = x^7 - 3.442165x^6 +$$
$$4.673633x^5 - 3.172047x^4 + 1.123507x^3$$
$$-0.196458x^2 + 0.0141174x - 0.000244$$
$$= (x-0.024679)(x-0.125167)$$
$$(x-0.286584)(x-0.472153)$$
$$(x-0.692886)(x-0.866876)$$
$$(x-0.973820).$$

Sixth Sequence.

$$p_0(x) = 1 \qquad p_1(x) = x - 0.38^{\bullet}$$
$$p_2(x) = x^2 - 0.913043x + 0.132850$$
$$= (x-0.181637)(x-0.731407)$$
$$p_3(x) = x^3 - 1.424936x^2$$
$$+0.529531x - 0.040511$$
$$= (x-0.102976)(x-0.452428)$$
$$(x-0.8695314)$$
$$p_4(x) = x^4 - 1.909026x^3 +$$
$$1.154159x^2 - 0.237349x + 0.010953$$
$$= (x-0.064058)(x-0.296254)$$
$$(x-0.624464)(x-0.924251)$$
$$p_5(x) = x^5 - 2.422255x^4 + 2.073462x^3$$
$$-0.743538x^2 + 0.100749x - 0.003172$$
$$= (x-0.044231)(x-0.215563)$$
$$(x-0.474160)(x-0.739349)$$
$$(x-0.948952)$$
$$p_6(x) = x^6 - 2.916365x^5 + 3.204091x^4$$
$$-1.641618x^3 + 0.391697x^2$$
$$-0.0372331x + 0.000842$$

$= (x-0.032057)(x-0.160228)$
$(x-0.352674)(x-0.589214)$
$(x-0.818132)(x-0.964061)$

$p_7(x) = x^7 - 3.415402x^6 + 4.598367x^5$
$- 3.092585x^4 + 1.084243x^3$
$- 0.187277x^2 + 0.013267x - 0.000226$
$= (x-0.024299)(x-0.123035)$
$(x-0.278772)(x-0.481014)(x-0.672450)$
$(x-0.862721)(x-0.973110).$

## 3. Example

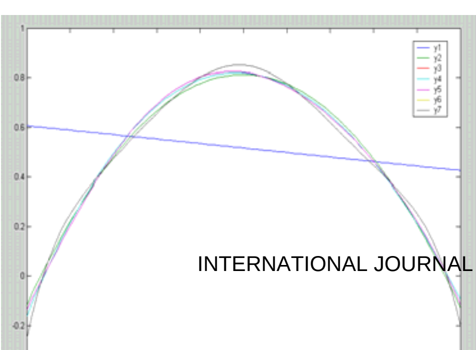
Examle 1.Use the first sequence and the fifth to smooth the curve

$$f(x) = \begin{cases} 2x, & x \in [0,0.5) \\ 2(1-x), & x \in [0.5,1] \end{cases}.$$

For the first weight function, we obtain.

$y_1 = 0.606061\text{-}0.181818\,x$
$y_2 = -0.116959+3.73099\,x - 3.74269\,x^2$
$y_3 = -0.1630689+4.251832\,x$
$-4.969608\,x^2 +0.7850438595\,x^3$
$y_4 = -0.16307+4.25189\,x - 4.969876\,x^2$
$+0.785447\,x^3 - 0.000196\,x^4$
$y_5 = -0.133721+3.404703\,x$
$+0.738025\,x^2 - 13.913479\,x^3$
$+16.065104\,x^4 - 6.277320\,x^5$
$y_6 = -0.251467 + 8.172574\,x$
$-45.30733\,x^2 + 164.657859\,x^3$
$-310.272999\,x^4 + 274.931115\,x^5$
$-92.132165\,x^6$
$y_7 = -0.251467 + 8.172574\,x$
$-45.307339\,x^2 + 164.657858\,x^3$
$-310.272900\,x^4 + 274.931115\,x^5$
$-92.132165\,x^6$

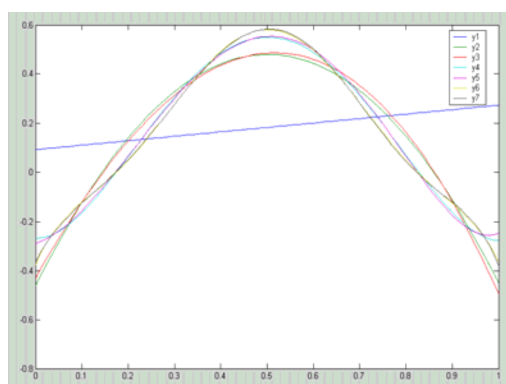
and their graphs are in the following figure.



For the fifth weight, we obtain.

$y_1 = 0.090909\text{-}0.181818\,x$
$y_2 = -0.461988+3.754386\,x - 3.74269\,x^2$
$y_3 = -0.429135+3.332253\,x$
$-2.614477\,x^2 - 0.785044\,x^3$
$y_4 = -0.267865 - 0.038765\,x$
$+13.180985\,x^2$
$+26.176532\,x^3 + 13.024469\,x^4$
$y_5 = -0.288722+0.613999\,x$
$+8.41654\,x^2 - 12.981746\,x^3$
$2.278116\,x^4 + 6.269652\,x^5$
$y_6 = -0.37415 + 4.316624\,x$
$-29.74765\,x^2 + 143.974119\,x^3$
$-303.819136\,x^4 + 277.191788\,x^5$
$-91.907466\,x^6$
$y_7 = -0.362424 + 3.639586\,x$
$-20.325445\,x^2 + 90.090233\,x^3$
$-151.686320\,x^4 + 53.04218\,x^5$
$+73.180292\,x^6 - 47.960457\,x^7$

and their graphs are in the following figure.



Example 2. Use the second sequence to smooth the curve

$$f(x)=\frac{2}{3}+3x-\frac{9}{2}x^2+\frac{3}{2}x^3,\quad x\in[0,1].$$

We obtain the following polynomials.

$y_1 = 1.177485380\text{-}0.1605263150\,x$

$y_2 = 0.74693+2.089474\,x$
$-2.249999991\,x^2$

$y_3 = 0.666667+3\,x$
$-4.499999526\,x^2+1.499999690\,x^3$

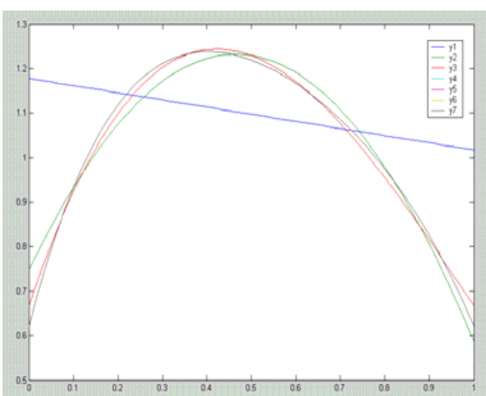
$y_4 = 0.620399+3.889095\,x$
$-8.369292\,x^2+7.460391240\,x^3$
$-2.980196\,x^4$

$y_5 = 0.6203997+3.88909\,x$
$-8.369258\,x^2+7.460306\,x^3$
$-2.9801\,x^4-0.000038\,x^5$

$y_6 = 0.6203987+3.889130\,x$
$-8.369652\,x^2+7.4618605\,x^3$
$-2.982995\,x^4+0.002502\,x^5$
$-0.000847\,x^6$

$y_7 = 0.620399+3.88913\,x-8.369652\,x^2$
$+7.46186\,x^3-2.982995\,x^4+0.0025024\,x^5$
$-0.000847\,x^6$

and their graphs are in the following figure.



Example 3. Use the third sequence to smooth the curve

$$f(x)=\frac{4}{3}-3x+\frac{9}{2}x^2-\frac{3}{2}x^3,\quad x\in[0,1].$$

We obtain the following polynomials.

$y_1 = 1.0250790.144286\,x$

$y_2 = 1.344524\text{-}2.105714\,x+2.25\,x^2$

$y_3 = 1.416667\text{-}3\,x+4.5\,x^2\text{-}1.5\,x^3$

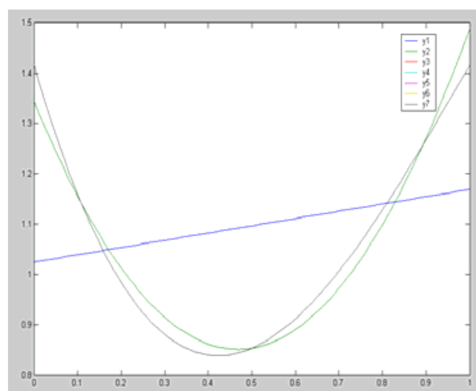
$y_4 = 1.416667\text{-}3\,x$
$+4.500011\,x^2-1.500017\,x^3$
$+0.000009\,x^4$

$y_5 = 1.416667\text{-}3.000008\,x$
$+4.500052\,x^2-1.500127\,x^3+0.000132\,x^4$
$-0.000050\,x^5$

$y_6 = 1.416668+3.000054\,x$
$+4.500532\,x^2-1.502111\,x^3+0.003911\,x^4$
$-0.003393\,x^5+0.0011145\,x^6$

$y_7 =$ 1.416656-
$2.999346\,x+4.490583\,x^2-1.445142\,x^3$ -
$0.155626\,x^4$
$+0.228489\,x^5-0.1669527\,x^6$
$+0.048019\,x^7$

and their graphs are in the following figure.



Example 4. Use the fourth sequence and the sixth sequence to smooth the curve of

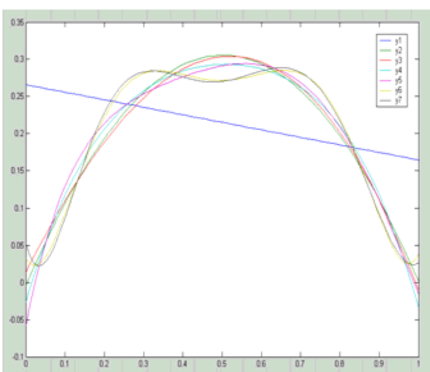
$$f(x)=\begin{cases} x, & x\in[0,\frac{1}{3}) \\ 3x^2-3x+1, & x\in[\frac{1}{3},\frac{2}{3}) \\ 1-x, & x\in[\frac{2}{3},1] \end{cases}.$$

For the fourth, we obtain.

$y_1 = 0.265700\text{-}0.101449\,x$

$y_2 = -0.004258+1.233512\,x$
$-1.228164\,x^2$
$y_3 = 0.014346+1.036197\,x$
$-0.770901\,x^2 -0.290314\,x^3$
$y_4 = -0.026417+1.784357\,x$
$-3.8756287\,x^2 +4.258768\,x^3$
$-2.175581\,x^4$
$y_5 = -0.057582+2.646024\,x$
$-9.479881\,x^2 +18.424777\,x^3$
$-.490017\,x^4 +5.941021\,x^5$
$y_6 = 0.032638-0.920027\,x$
$+24.220864\,x^2 -110.002361\,x^3$
$+213.825647\,x^4 -190.978059\,x^5$
$+63.859403\,x^6$
$y_7 = 0.052091-1.951126\,x$
$+37.411191\,x^2 -$
$179.802678\,x^3 +398.090969\,x^4 -$
$447.464946\,x^5$
$+243.9241265\,x^6 -50.23289231\,x^7$

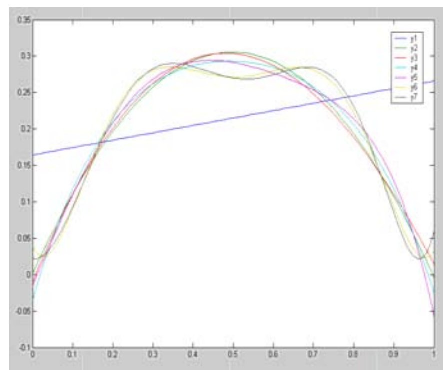
and graphs are in the following figure.



For the sixth sequence, we obtain.

$y_1 = 0.164251+0.101449\,x$
$y_2 = 0.001089+1.222816\,x -1.228164\,x^2$
$y_3 = -0.010672+1.376546\,x$
$-1.641842\,x^2 +0.2903137527\,x^3$
$y_4 = -0.034500+1.892905\,x$
$-4.152748\,x^2 +4.443455933\,x^3$
$- 2.175529033\,x^4$
$y_5 = -0.015653+1.294255\,x$
$+0.265344\,x^2 -7.877027\,x^3$
$+12.217483\,x^4 -5.941988\,x^5$
$y_6 = 0.038156-1.085574\,x$
$+25.301366\,x^2 +217.012575\,x^4$
$-112.803964\,x^3 -192.346605\,x^5$
$+63.916762\,x^6$
$y_7 = 0.022062-0.142082\,x$
$+11.9831834\,x^2 -35.698000\,x^3$
$-2.916682\,x^4 +134.666381\,x^5$
$-178.969629\,x^6 +71.115022\,x^7$

and their graphs are in the following figure.



## 4. Conclusion

All results are as good as they are expected. We recommend all six sequences of orthogonal polynomials

## References

[1] Maitree Podisuk, Pongpan Rattanathanawan and Pimpak Phataranavik, Sequences of Orthogonal Polynomials with Step Functions as Their Weight Functions, Proc. of the 8th WSEAS Int. Conference on Mathematical Method and Computational Techniques in Electrical Engineering, Bucharest, Romania, October 16-18, 2006.