Modern mathematical model using mixed p-z approach for the analyze of the switching power circuits

Jiri Klima

Abstract— The mathematical method for the time-domain analysis of power converters with periodic pulse width modulation (PWM) is developed .The method is based on mixed p-z description of linear periodically time-varying system. The mathematical model uses the Laplace and modified Z transforms. The solution is not dependent on the number of the pulses of the PWM pattern. Instead of solution of algebraic equations the change of switching instants is reflected in the solution only by a change in two values mk and nk. All the results were visualized from the derived equations by the programme Mathcad. The derived equations are validated using a 3 kW three-phase inverter

Keywords— Modified Z-transform, Laplace transform, switching circuits

I. Introduction

Several methods have been presented for the time analysis of linear circuits containing periodically operated switches in electronic opened-loop systems [1], [2] [3].However, the approach used in these methods depends heavily on matrix manipulations as they require matrix inversion as well as exponentiation. Besides, it requires solution of many algebraic equations.

Many electronic systems such as the inverters with Pulse Width Modulation (PWM) can be modeled with periodically varying parameters. Recent developments in high switching frequency power devices, such as IGBT, offer the possibility of developing high frequency PWM control techniques. Voltage waveforms of such modulated inverters contain many pulses and gaps. It is important to known current response for such complicated voltage waveforms for a proper design.

This paper brings a mathematical model which uses the Laplace and modified Z- transform (mixed p-z approach).The model enables one to determine both transient and steady state response in a relatively simple and lucid form. Method for finding the Laplace transform of the voltage vector is also presented. The solution is not dependent on the number of the pulses of the PWM pattern.

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J. Klima is with rhe department of Electrical Engineering and Automation,Faculty of Engineering,University of Life Science in Prague,16521,Prague6-Suchdol,CzechRepublic,(phone:420224383203;e-mail:klima@tf.czu.cz).

The change of the switching instants is reflected in the solution by a change in only two values that determines transport delay .If we compare the proposed method with existing techniques the main advantages can be found as follows:

-The solution is in an analytical closed-form, which does not require matrix inversion and exponentiation.

-An analytical solution, contains only one equation for the currents and equation for two values describing the solution in prepulse, inside-pulse and postpulse time, and does not require solving many algebraic equations as in existing methods. It means that for output PWM waveforms the model makes it applicable irrespective of the number of pulses per output waveform.

-From the analytical equation we can easily derive the characteristic values of the inverter or of the motor, such as, the peak, mean and rms values, both in the transient steady states.

As it was mentioned before, the model is applicable for the time response in opened-loop time-variable circuits, as it requires an explicit form of the output voltage of the converters.

II. Mathematical Model

The energy conversion of many power electronics converters is achieved by cyclically controlled switching topological configurations. Let us consider power electronic circuit with an output voltage of the form given by the equation (1).These voltage waveforms is typical for DC-DC converters.

$$v(t)=\begin{cases} V_{dc} & \text{for} \quad nT+T_{kA}\le t< nT+T_{kB} \\ 0 & \text{for} \quad nT+T_{kB}\le t< nT+T_{(k+1)A} \end{cases} \tag{1}$$

where k , and n are integers , that means number of the pulse used inside of period, and number of period, respectively. T is a period, T_{kA} and T_{kB} are start point setting time and end point setting time, respectively.

Let us express time as

$$t=(n+\varepsilon)T , \quad n=0,1,2,... \quad , 0<\varepsilon\le 1 \tag{2}$$

then (1) can be expressed in per unit time

$$v(t)=\begin{cases} V_{dc} & \text{for} \quad \varepsilon_{kA}\le\varepsilon<\varepsilon_{kB} \\ 0 & \text{for} \quad \varepsilon_{kB}\le\varepsilon<1 \end{cases} \qquad (3)$$

From the definition of the Laplace transform of the periodic signal, we can find the Laplace transform of the voltage v(t):

$$V(p)=\frac{1}{(1-e^{-pT})}\int_0^T v(t)e^{-pt}dt=$$

$$V_{dc}\frac{1}{p}\left(\frac{e^{pT}}{e^{pT}-1}\right)\sum_{k=1}^{M}\left(e^{-pT\varepsilon_{kA}}-e^{-pT\varepsilon_{kB}}\right) \qquad (4)$$

M is number of pulses per period T.

But in a more complex circuits containing the period pulse width modulation (PWM) we can derive the Laplace transform from the relation between the Laplace and Modified Z-Transform as follows:

$$V(p)=\begin{matrix}\displaystyle\sum_{n=0}^{\infty}\int_0^1 v(n,\varepsilon)e^{-p(n+\varepsilon)T}Td\varepsilon= \\ \displaystyle T\int_0^1 V(z,\varepsilon)e^{-pT.\varepsilon}d\varepsilon\end{matrix} \qquad (5)$$

Where

$z=e^{pT}$,

$V(z,\varepsilon)$ is the modified Z transform of $v(n,\varepsilon)$ [7],[8] defined by:

$$V(z,\varepsilon)=\sum_{n=0}^{\infty}v(n,\varepsilon)z^{-n} \qquad (6)$$

For the voltage given by (3) we have

$$v(n,\varepsilon)=1^n\sum_k f(\varepsilon,k) \qquad (7)$$

where $f(\varepsilon,k)$ is a switching given as (3)

and the modified Z-transform of (7) is

$$V(z,\varepsilon)=\frac{z}{z+1}\sum_k f(\varepsilon,k) \qquad (8)$$

Using (5) and (7) we can again derive the Laplace transform of the voltage as in (4).

Now, we suppose that voltage with the Laplace transform $V(p)$ is feeding load with admittance:

$$Y(p)=\frac{A(p)}{B(p)}=\sum_{s=1}^{L_S}\frac{A(p_S)}{B'(p_S)}\frac{1}{p-p_S} \qquad (9)$$

where: $B'(p_S)=\left[\frac{dB}{dp}\right]_{p=p_S}$

p_s are roots of the characteristic equation:

$$B(p)=0 \qquad (10)$$

L_s is a order of the polynomial B(p).

Thus, using (4) and (9) the Laplace transform of the load current can be expressed as:

$$V_{dc}\frac{1}{p}\left(\frac{e^{pT}}{e^{pT}-1}\right)\sum_{k=1}^{M}\left(e^{-pT\varepsilon_{kA}}-e^{-pT\varepsilon_{kB}}\right)\frac{A(p)}{B(p)}=$$

$$R(e^{pT})Q(p) \qquad (11)$$

where:

$$R(e^{pT})=\frac{e^{pT}}{e^{pT}-1} \qquad (11a)$$

$$Q(p)=\frac{V_{dc}}{p}\frac{A(p)}{B(p)}\sum_{k=1}^{M}\left(e^{-pT\varepsilon_{kA}}-e^{-pT\varepsilon_{kB}}\right) \qquad (11b)$$

As can be seen from (11), the Laplace transform of the current consists of two multiplicative parts. One (R (e^{pT})) is a function of e^{pT} -operator, the other (Q (p)) is a function of p-operator. To find original function of (11) we can use the residual theorem.

But the inverse transform of (11) can not be carried out in direct way as it contains infinite number of poles given by

$$e^{pT}-1=0. \qquad (12)$$

From (11) it may be seen that both polynomials can be separated into two multiple parts and so we can transform (11) into the modified Z transform [8].If doing so, we get in the Z-space:

$$I(z,\varepsilon)=R(z).Z_m\{Q(p)\} \qquad (13)$$

with $Z_m\{\ \}$ denoting the modified Z transform operator.

In order to find Z transform of Q(p) we must use the translation theorem in Z transform which holds:

$$Z_m\{e^{-p.a}.F(p)\}=z^{-x}.F(z,\varepsilon-a+x) \qquad (14)$$

where parameter x is given by :

$$x=\begin{cases} 1 & \text{for} \quad 0\le\varepsilon<a \\ 0 & \text{for} \quad a\le\varepsilon<1 \end{cases} \qquad (15)$$

If we want to express translation for k-th pulse, with the beginning ε_{kA} and the end ε_{kB},

(pulse-width $\Delta\varepsilon_k=\varepsilon_{kB}-\varepsilon_{kA}$) we can use two parameters, namely m_k and n_k to determine per unit time for prepulse,inside-pulse and postpulse,respectively.

m_k is a parameter that defines the beginning of k-the pulse ε_{kA}, n_k is a parameter that defines the end of the k-pulse ε_{kB} .According to (15) we can write for m_k and n_k, respectively:

$$m_k=\begin{cases} 1 & \text{for } 0\le\varepsilon<\varepsilon_{kB} \\ 0 & \text{for } \varepsilon_{kA}\le\varepsilon<1 \end{cases} \qquad n_k=\begin{cases} 1 & \text{for } 0\le\varepsilon<\varepsilon_{kA} \\ 0 & \text{for } \varepsilon_{kB}\le\varepsilon<1 \end{cases} \qquad (16)$$

By means of these two parameters we can express per unit time for the three intervals:

a) $0<\varepsilon \le \varepsilon_{kA}$ prepulse per unit time. $m_k=1, n_k=1$

b) $\varepsilon_{kA} < \varepsilon \le \varepsilon_{kB}$ inside pulse per unit time .$m_k=0, n_k=1$ (17)

c) $\varepsilon_{kB} < \varepsilon \le 1$ postpulse per unit time .$m_k=0, n_k=0$

Thus, in the period nT, for per unit time $0<\varepsilon\le 1$, we obtain from (17) two parameters m_k and n_k, that will be used for solution in Z transform.

Using (9) and Heaviside theorem in (11b):

$$\frac{A(p)}{pB(p)}=\frac{1}{p}\frac{A(0)}{B(0)}+\sum_{s=1}^{L_s}\frac{A(p_s)}{p_s B'(p_s)}\frac{1}{(p-p_s)} \qquad (18)$$

we can find the original of (13) by the definition of the inverse Z transform:

$$I(n,\varepsilon)=\frac{1}{2\pi j}\oint I(z,\varepsilon)z^{n-1}dz \qquad (19)$$

An integral (19) may be solved by means of the residual theorem

III. THREE-PHASE VOLTAGE SOURCE INVERTER WITH SPACE-VECTOR PWM

In that section we investigate the three-phase half-bridge voltage source inverter feeding a balanced three-phase Y-connected load.

Generally, the three output voltage variables $v_{in}(t)$, i=a,b,c can be projected into two variables, in the complex plane α and β using the following transformation:

$$\mathbf{V}(t)=\frac{2}{3}\left[\, v_{an}(t)+a.v_{bn}(t)+a^2.v_{cn}(t)\,\right]=$$

$$V_\alpha(t)+jV_\beta(t), \qquad a=e^{j\frac{2\pi}{3}}, \qquad (20)$$

The three-phase voltage source inverter has eight discrete voltage vectors in the complex plane as indicated in Fig.1, $\mathbf{V}_1$ through $\mathbf{V}_6$, with length $2V_{dc}/3$ and two zero vectors,$\mathbf{V}_0$ and $\mathbf{V}_7$ (connecting all of the three-phase of the load to positive or negative rail of the DC bus).

From the mathematical point of view both zero vectors have the same effect:

$$\mathbf{V}_0=\mathbf{V}_7=0 \qquad (20a)$$

By substituting the phase voltages for each switching state into (20), the following discrete space vectors are obtained:

$$\mathbf{V}(n)=\frac{2V_{dc}}{3}e^{jn\pi/3}, \quad n=0,1,2\ldots\ldots \qquad (21)$$

These vectors thus form vertices of hexagon as shown in Fig.1.

As was mentioned, more vectors within sampling period are utilized in case of modulation .At present, one of the most modern modulation method is Space Vector Pulsewidth Modulation. (SVM).As the synchronous SVM modulation is a periodical with T, the voltage vector can be expressed, in n-th sector, as

$$\mathbf{V}(n,\varepsilon)=\sum_{k=1}^{M}\frac{2V_{dc}}{3}e^{j\pi n/3}f(\varepsilon,k)e^{j\pi\alpha(k)/3} \qquad (22)$$

From (22) it can be seen, that all vectors are rotated in the next sector through $\pi/3$,and in each sector are vectors modulated with time dependency given by $f(\varepsilon,k)$, and also with the angle dependency given by $e^{j\pi\alpha(k)/3}$.

M is number of the vectors, which are used within a sector period T. $\alpha(k)$ defines the sequence of the phase shift of the vectors, and for SVM with two adjacent vectors has value 1 or 0 .As was mentioned, in the employed, synchronous sampling mode, the cycle of the output frequency in the vector space is divided into six 60^o wide sectors and each sector into N_1 segments representing individual sampling interval. In the SVM strategy, the inverter state is changed three times within each sampling interval. For instance, we can use sequence of the vectors in the first sector:

$\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2$

It means, that $\alpha(0)=0$, $\alpha(1)=1$.Two adjacent vectors with the angles, are used:

I) zero vector $\mathbf{V}_0$
ii) Vector $\mathbf{V}_1$ with angle: $\alpha(0)\pi/3=0$, (real axis)
iii) Vector $\mathbf{V_2}$ with angle: $\alpha(1)\pi/3=\pi/3$.

For practical purpose, the sequence of pulses and gaps defined for the sector T is stored in the microcomputer memory. Each sector is further divide into N_1 segments, which form sampling interval. Duration of the individual states is determined from simple formulas. From the angle Point of view, the complex plane of the voltage vectors of the inverter is divided into six 60^0 wide sectors (0^0-60^0,60^0-120^0, etc.)In the subsequent sixth of the period the direction of the voltage vectors is rotated through $\pi/3$.It means, that this modulation is a *periodical with a period T.* Fig.1 clearly shows that in the first sector ,the mean value of the voltage vector $\mathbf{V}_{AV}$ can be calculated using the relation:

$$e^{j\rho}\mathbf{V}_{AV}\frac{\Delta T}{T}=\mathbf{V}_1\frac{\Delta T_1}{T}+\mathbf{V}_2\frac{\Delta T_2}{T}$$
$$\Delta T=\Delta T_1+\Delta T_2+\Delta T_0 \qquad (23)$$

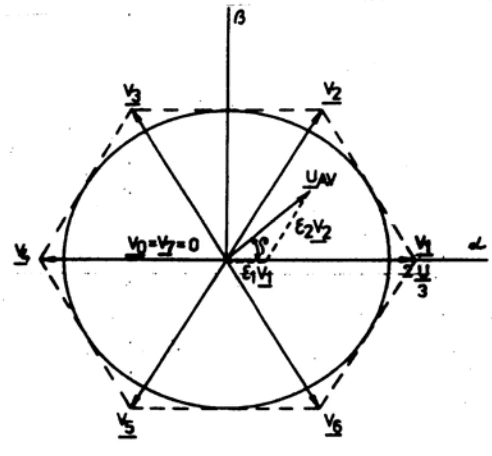

Fig.1 Voltage space-vector representation in αβ complex plane

where ΔT_1 is dwell time of vector $\mathbf{V}_1$, ΔT_2 is dwell time of vector $\mathbf{V}_2$, and ΔT_0 is dwell time of vector $\mathbf{V}_0$, or $\mathbf{V}_7$.

ΔT is a sampling interval.

$$\Delta T = T/N1 \tag{24}$$

ρ is an angle that defines position of the reference vector $\mathbf{V}_{AV}$ with respect to real axis in complex αβ plane

If we express vectors $\mathbf{V}_1, \mathbf{V}_2$ in stator co-ordinate system, we get:

$$\mathbf{V}_1 = V_1 e^{j0} = \frac{2V_{dc}}{3} e^{j0} \tag{25a}$$

$$\mathbf{V}_2 = V_2 e^{j60^0} = \frac{2V_{dc}}{3} e^{j60^0} \tag{25b}$$

By substituting (25a) and (25b) into (23) and solving it for the real and imaginary axis we get:

$$\Delta\varepsilon_1 = \Delta T_1/T = \varepsilon_{1B} - \varepsilon_{1A} = g \sin(60^0 - \rho)/N_1$$

$$\Delta\varepsilon_2 = \Delta T_2/T = \varepsilon_{2B} - \varepsilon_{2A} = g \sin\rho / N_1 \tag{26}$$

$$\Delta\varepsilon_0 = \Delta T_0/T = 1/N_1 - g \sin(60^0 + \rho)/N_1$$

ε_{1A} and ε_{1B} are respectively, the beginning and end of duration of vector $\mathbf{V}_1$, ε_{2A} and ε_{2B} are respectively, the beginning and end of duration of vector $\mathbf{V}_2$. $\varepsilon_1, \varepsilon_2$ and ε_0 are respectively, per unit dwell times (duty ratios) of the applied vectors .

$$g = \frac{\sqrt{3} V_{AV}}{V_{dc}} \tag{27}$$

G is the transformation (modulation) factor,

V_{dc} is the voltage of DC bus.

With regard to SVM strategy mentioned, we get from (5) and (22) the Laplace transform for the stator voltage space-vector:

$$\mathbf{V}(p) = \frac{2V_{dc}}{3} \frac{1}{p} \left(\frac{e^{pT}}{e^{pT} - e^{j\pi/3}} \right) \sum_{k=1}^{M} e^{j\pi\alpha(k)/3} \left(e^{-pT\varepsilon_{kA}} - e^{-pT\varepsilon_{kB}} \right) \tag{28}$$

where $\varepsilon_{kA}T$ and $\varepsilon_{kB}T$ are respectively, the beginning and the end of application of k-th non-zero vector.

Again, we suppose that voltage with the Laplace transform $\mathbf{V}(p)$ is feeding load with admittance (9)

Using (9) and (28) the Laplace transform of the space vector of the load current can be expressed as follows:

$$\mathbf{I}(e^{pT}, p) = \mathbf{V}(p) Y(p) = R(e^{pT}) Q(p) \tag{29}$$

Again, the Laplace transform of the current vector consists of two multiplicative parts. One ($R(e^{pT})$) is a function of z-operator, the other ($Q(p)$) is a function of p-operator . Comparing (29) with (11a) and (11b) one obtains:

$$R(e^{pT}) = \frac{e^{pT}}{e^{pT} - e^{j\pi/3}} \tag{30}$$

$$Q(p) = \frac{2V_{dc}}{3p} \frac{A(p)}{B(p)} \sum_{k=1}^{M} e^{j\pi\alpha(k)/3} \left(e^{-pT\varepsilon_{kA}} - e^{-pT\varepsilon_{kB}} \right) \tag{31}$$

By transforming (29) into modified z -space we get:

$$\mathbf{I}(z, \varepsilon) = R(z).Z_m\{Q(p)\} \tag{32}$$

with $Z_m\{\ \}$ denoting the modified Z transform operator.

Using parameters m_k, n_k in (17), and Heaviside theorem (11),we can express (32) in the modified Z-space as follows:

$$I(z,\varepsilon) = \frac{2V_{dc}}{3} \frac{z}{z - e^{\frac{j\pi}{3}}} \sum_{k=1}^{M} \left\{ \begin{array}{l} \frac{A(0)}{B(0)} e^{\frac{j\pi\alpha(k)}{3}} \frac{z}{z-1} \left(z^{-m_k} - z^{-n_k} \right) + \\ \sum_{s=1}^{L_s} \frac{A(p_s)}{p_s B'(p_s)} e^{\frac{j\pi\alpha(k)}{3}} \frac{z e^{p_s T\varepsilon}}{z - e^{p_s T}} \\ \left[z^{-m_k} e^{p_s T(m_k - \varepsilon_{kA})} - z^{-n_k} e^{p_s T(n_k - \varepsilon_{kB})} \right] \end{array} \right\} \tag{33}$$

Equation (33) has simple poles $e^{j\pi/3}$, 1 ,$e^{p_s T}$.The inverse Z transform of (33) can be found using the residua theorem .If doing so, we can express the time dependency of the load current by the following formula:

Equation (33) is the time dependency of the current space vector in the stator co-ordinate system .As can be seen, it is in closed-form. For concrete solution we must substitute into (33) only parameters of the load ($A(p)$,$B(p)$) and parameters of the inverter (V_{dc},ε_{kA},ε_{kB},$\alpha(k)$).The solution contains two portions.

Since p_s includes a negative real part (we consider stable systems), the second portion of (33) consisting $e^{p_s T(n+\varepsilon)}$ attenuates, for $n \to \infty$, forming the transient component of the current space vector $\mathbf{i}_T(n,\varepsilon)$.

$$\mathbf{i}(n,\varepsilon)=\left[\begin{array}{l}\dfrac{A(0)(e^{-j\pi m_k/3}-e^{-j\pi n_k/3})}{B(0)(e^{j\pi/3}-1)}+\\ \displaystyle\sum_{s=1}^{L_s}\frac{A(p_s)e^{p_s T.\varepsilon}}{p_s.B'(p_s)(e^{j\pi/3}-e^{p_s T})}\\ (e^{-j\pi m/3+p_s T(m_k-\varepsilon_{kA})}-e^{-j\pi n_k/3+p_s T(n_k-\varepsilon_{kB})}\end{array}\right]$$

$$e^{j\pi/3(n+1)}+$$

$$+\sum_{k=1}^{M}\left\{\frac{2V_{dc}}{3}e^{j\pi\alpha(k)/3}\sum_{s=1}^{L_s}\frac{A(p_s)}{p_s B'(p_s)}\frac{e^{p_s T}}{(e^{j\pi/3}-e^{p_s T})}(e^{-p_s T\varepsilon_{kB}}-e^{-p_s T\varepsilon_{kA}})e^{p_s T(n+\varepsilon)}\right\}$$

$$=\mathbf{i}_S(n,\varepsilon)+\mathbf{i}_T(n,\varepsilon) \qquad (34)$$

The term

$$e^{j\pi(n+1)/3}=\cos\pi(n+1)/3+j.\sin\pi(n+1)/3 \qquad (35)$$

therefore, the first portion of (34) is the steady-state component of the current space vector $\mathbf{i}_s(n,\varepsilon)$.
As it was mentioned before we consider three-phase R,L series load. Equation (34) thus has only one simple root:

$$p_1=\frac{-R}{L} \qquad (36)$$

By substituting p_1 into (34) we can write for the load current components:

a) steady-state component

$$\mathbf{i}_S(n,\varepsilon)=\sum_{k=1}^{M}\frac{2V_{dc}}{3}e^{j\pi\alpha_k/3}\frac{1}{R}\left\{\begin{array}{l}\left(\dfrac{e^{-j\pi m_k/3}-e^{-j\pi n_k/3}}{e^{j\pi/3}-1}\right)-\\ \left(\dfrac{e^{-RT\varepsilon/L}}{e^{j\pi/3}-e^{-RT/L}}\right)(e^{-j\pi m_k/3-RT(m_k-\varepsilon_{kA})/L}-\\ e^{-j\pi n_k/3-RT(n_k-\varepsilon_{kB})/L})e^{j\pi(n+1)/3}\end{array}\right\} \qquad (37a)$$

b) *transient component*

$$\mathbf{i}_T(n,\varepsilon)=\sum_{k=1}^{M}\left\{\frac{2V_{dc}}{3}e^{j\pi\alpha_k/3}\frac{1}{R}e^{-RT/L}\left(\frac{e^{RT\varepsilon_{kA}/L}-e^{RT\varepsilon_{kB}/L}}{e^{j\pi/3}-e^{-RT/L}}\right)e^{-RT(n+\varepsilon)/L}\right\} \qquad (37b)$$

m_k,n_k are given by (16).

As graphical examples, we can see in the following figures some analytical results. The graphical waveforms were visualized from the derived equations by the Programme MATCAD.
The parameters for the examples are as follows:
SVM - Number of segments $N_1=2$, modulation factor g=0.2.Output frequency of the inverter is: f_1=50Hz.
A three-phase static inductive load has the parameters: R=623Ω ,ω_1L=502Ω.

Fig.2 shows the phase A voltage given by real part of (22).The voltages in other phases are shifted by the angles ±2π/3, respectively.

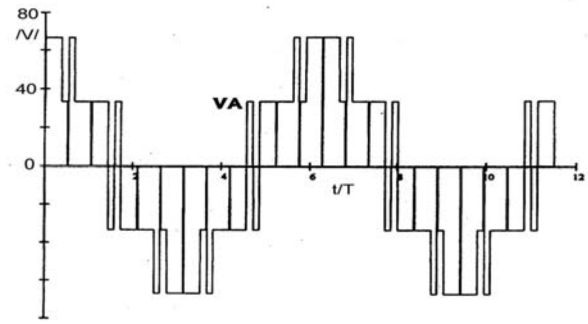

Fig.2 Load voltage in phase A

Fig.3a shows trajectory of the steady-state current vector in complex plane. This trajectory is given by (36).Fig.3b shows the phase A steady-state current given by the real part of previous Figure .Fig.4 shows the phase A transient current, given by the real part of (37), and Fig.5 shows trajectory of the overall load current again in the phase A, for the same parameters .Again, currents in other phases are shifted by ±2π/3, .respectively. From (36) we can derive easily the solution for six-step waveform (without modulation).

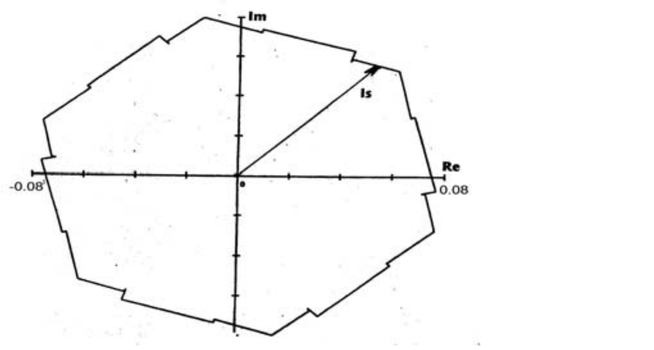

Fig.3a Stator steady-state current trajectory in αβ complex plane

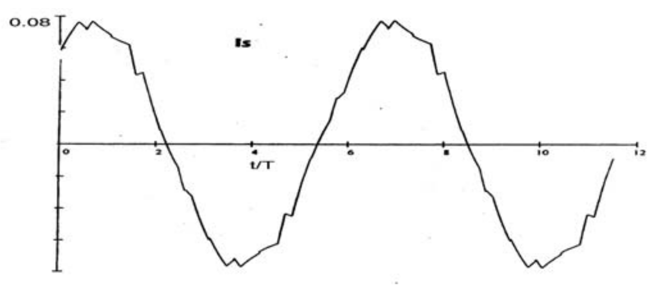

Fig.3b. Stator steady-state current in phase A

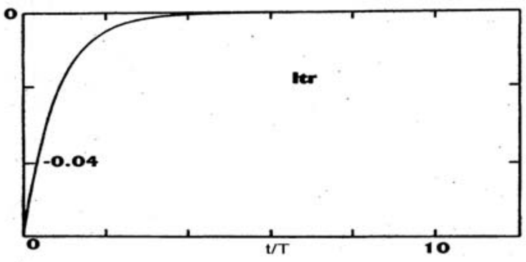

Fig.4 .Transient current in phase A

Six-Step waveforms:

In Eq (36), which is valid for the steady-state, we substitute:
M=1 (one pulse per sector) $\varepsilon_{1A}=0, \varepsilon_{1B}=1, m_1=0, n_1=1$.
By substituting these values into (36) we obtain for the steady-state vector current of the RL load very simple equation::

$$\mathbf{i}_S(n,\varepsilon)= \frac{2V_{dc}}{3R} e^{\frac{j\pi n}{3}} \left[1-e^{\frac{-RT}{L}\varepsilon} \left(\frac{e^{\frac{j\pi}{3}}-1}{e^{\frac{j\pi}{3}}-e^{\frac{-RT}{L}}} \right) \right] \quad (38)$$

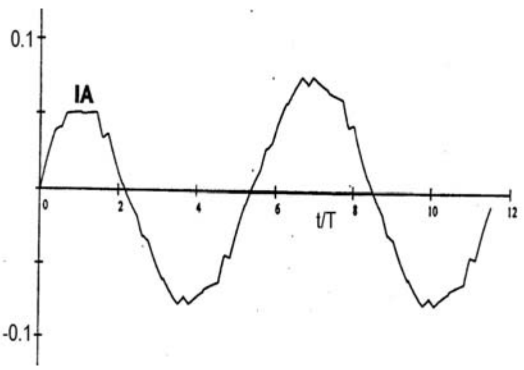

Fig.5 Overall current waveform in phase A

Putting n=0 and $0<\varepsilon\leq 1$, we get solution for the first sixth of the period, for n=1 and $0<\varepsilon\leq 1$, we get solution for the second sixth of the period, etc.
The A-phase current is given by real part of (38)

$$i_A(n,\varepsilon)=\mathrm{Re}\{\, \mathbf{i}_S(n,\varepsilon)\} \quad (39)$$

and is shown in Fig.6
For the voltage vector with six-step waveform we can write:

$$V(n,\varepsilon) = 2/3V_{dc}e^{jn\frac{\pi}{3}} = V(n) \quad (40)$$

For example A phase voltage is given by a real part:

$$v_A(n,\varepsilon)=v_A(n) = \mathrm{Re}\{\mathbf{V}(n,\varepsilon)\}=$$

$$2/3(V_{dc}.\cos\pi.n/3) \quad (41)$$

and is also shown in Fig.6 (dashed line)

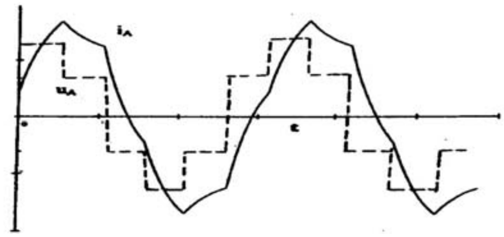

Fig.6 Six-step voltage and current waveform

To validate the performance of the mathematical model, the steady-state waveforms in [9] were obtained by numerically integrating the differential equations of the system starting from zero initial values of the currents. After the steady-state current waveforms are reached, the results obtained from the numerical solution are then compared with the waveforms of the current obtained from the analytical solution. The results from the numerical solution are identical with the results obtained from the analytical solution presented in [9] and also are presented in the paper.

IV. Frequency-Domain Analysis

A. Fourier series for the stator voltage vectors

We shall calculate the Fourier series of the periodic variation of the stator voltage space vector [6]:

$$\mathbf{V}(n,\varepsilon) = \sum_{k=-\infty}^{\infty}\left[\, \mathbf{C}_k e^{(jk\omega_1(n+\varepsilon)T} \,\right] \quad (42)$$

where $\omega_1 = 2\pi/T_1$ is the angular frequency of the fundamental harmonic. From (42) , the phase voltages can be expressed as:

$$\mathbf{v}_{An}(n,\varepsilon) = \mathbf{Re}\left\{ \sum_{\nu=-\infty}^{\infty} (C_\nu \mathbf{e}^{\mathbf{j}\nu\omega_1(\mathbf{n}+\varepsilon)\mathbf{T}}) \right\} \quad (43a)$$

$$v_{Bn}(n,\varepsilon) = \mathbf{Re}\left\{ e^{j4\pi/3} \sum_{\nu=-\infty}^{\infty} (C_\nu \mathbf{e}^{\mathbf{j}\nu\omega_1(\mathbf{n}+\varepsilon)\mathbf{T}}) \right\}$$

(43b)

$$\mathbf{v}_{Cn}(n,\varepsilon) = \mathbf{Re}\left\{ e^{j2\pi/3} \sum_{\nu=-\infty}^{\infty} (C_\nu \mathbf{e}^{\mathbf{j}\nu\omega_1(\mathbf{n}+\varepsilon)\mathbf{T}}) \right\} \quad (43c)$$

To derive the coefficients of the Fourier series, we can use the relationship between the Laplace transform of the periodic waveform and Fourier coefficients:

$$C_k = \left\langle \frac{1}{T_1}\left[\; 1-(e^{-pT_1})\, V(p) \;\right\rangle_{p=jk\omega_1} \quad (44)$$

$\mathbf{V}(p)$ is given by (28).

By substituting (28) into (44) we obtain the Fourier coefficients as follows:

$$\mathbf{C}_{\nu} = \mathbf{C}_{(1+6\nu)} = \\ = \frac{2V_{dc}}{3\pi j(1+6\nu)} \sum_{k=1}^{M} e^{(j\pi\alpha(k)/3)} \left[e^{\left(-j(1+6\nu)\frac{\pi}{3}\varepsilon_{kA}\right)} - e^{\left(-j(1+6\nu)\frac{\pi}{3}\varepsilon_{kB}\right)} \right] \qquad (45)$$

where

$$\nu = 0, \pm 1, \pm 2, .. \qquad (46)$$

B. Fourier series for the phase voltages

From voltage-space expression (45) we obtain the phase voltages as a real part of the complex equation (45) as:

$$v_{An}(n,\varepsilon) = \sum_{\nu=-\infty}^{\infty} \left[\frac{2V_{dc}}{\pi 3(1+6\nu)} \sum_{k=1}^{M} \left[\sin\frac{\pi}{6}(1+6\upsilon)\Delta\varepsilon_k \right] \sin\left[(1+6\nu) \left[\omega_1(n+\varepsilon)T - \frac{\pi}{6}(\varepsilon_{kA}+\varepsilon_{kB}) \right] + \frac{\pi\alpha(k)}{3} - 2\pi/3 \right] \right] \qquad (47)$$

As an example we can see from Fig.7 the Fourier approximation of the voltage space-vector with space-vector modulation. We take into account first 10 harmonics

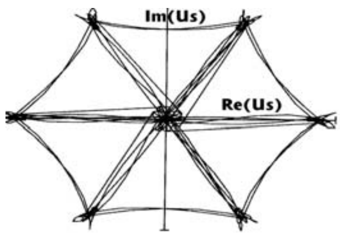

Fig.7 Fourier series approximation of the voltage space vector

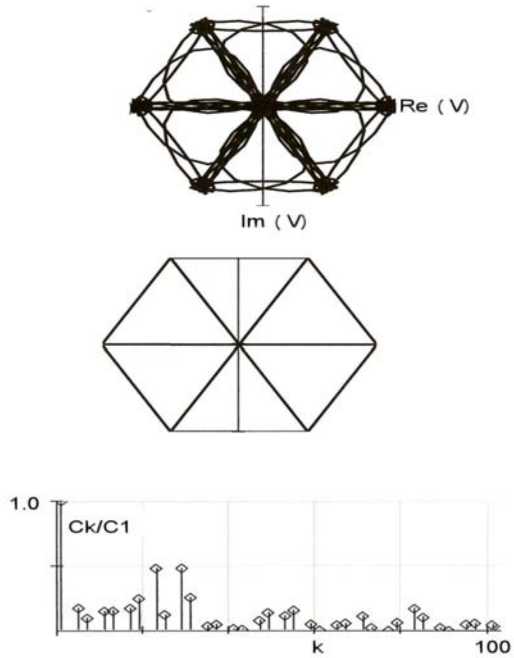

Fig.8 Voltage space vectors and harmonic spectrum. g=0.3,N_1=4 (f_{SW}=1200 Hz) ,f_1=50 Hz. Top: Fourier series approximation of voltage space vectors for ν=20.Middle: Ideal voltage space vectors. Bottom: Harmonic voltage spectrum.

From Fig.8 we can see the Fourier series approximation of the voltage space-vector (upper trace); ideal trajectory (middle trace) and Fourier spectrum (bottom trace) again for the space-vector PWM modulated voltage.

V. EXPERIMENTAL VERIFICATION

Validation of the derived analytical equations was also carried out using measurements with a 3 kW three-phase inverter supplying 2.7 kW cage-rotor induction motor 380V, 7.6A, and 1475 r/min. An IGBT inverter utilized Space Vector PWM with sampling intervals N_1=7, modulation factor g=0, 4, and with a fundamental frequency of the output voltage of 50 Hz.

Fig.8 shows experimental waveforms of the phase A steady-state load current (upper trace) and the phase A load voltage (lower trace).The corresponding theoretical phase A steady-state current given from (33) and phase voltage given from (25) are shown in Fig.9.As can be seen, there is very good agreement between measured and theoretical results, with correlation being better than 5% over most of the load range.

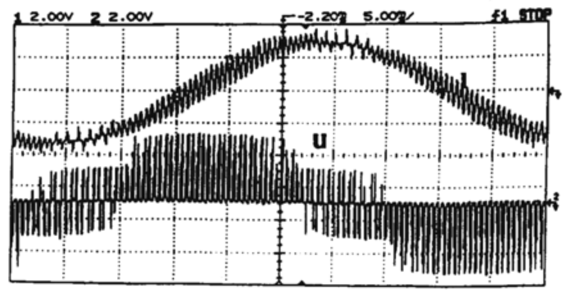

Fig.8 Experimental waveform of the stator voltage and current in phase

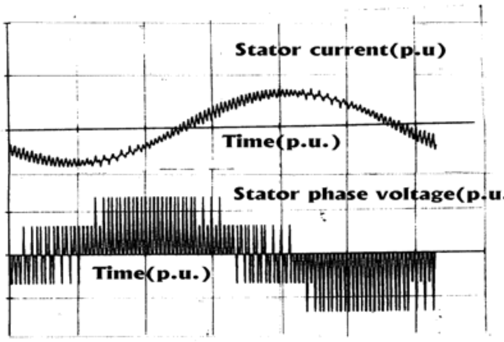

Fig.9 Analytical calculated waveform of the voltage and current in phase A.

VI Conclusion

An approach for the analysis of linear system containing periodically operated switches is described. The approach was demonstrated for DC-DC converter, three-phase voltage source inverter with Space Vector PWM and single-phase voltage source inverter, but it is applicable for all types of converters with explicitly determined output voltage. The mathematical model uses the Laplace and modified Z transforms. The steady -state and transient components of the load current are determined in a simple and lucid form that it avoids involved matrix inversion as well as exponentiation. All the results were visualized from the derived equations by the programme Mathcad. Experimental results prove the feasibility of the proposed mathematical model as compared with the conventional methods. Correlation between measurements and calculations is very good.

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Jiri Klima, He received his MSc degree from the Faculty of Electrical engineering of the Czech Technical University in Prague in 1968, and PhD degree he received in 1978.Since 1970 he has been in the Department of Electrical engineering and Automation, Technical Faculty of Czech University of Agriculture in Prague. In 1993 he defended habilitation on the theme" Mathematical models of electrical drives with semiconductor devices" and in 1993 he was appointed professor for the field of Electrotechnics and electronics. Since 2003 he is a dean of the Technical Faculty.His main field of interest is the mathematical models of electric circuits with the discrete components, electrical drives and power electronics. He is author or co-author of more than 250 technical papers from that field including papers published in IEEE and IEE Journals.