On the Enhanced Hyper-hamiltonian Laceability of Hypercubes

Tsung-Han Tsai, Tzu-Liang Kung, Jimmy J. M. Tan, and Lih-Hsing Hsu

***Abstract*—** A bipartite graph is hamiltonian laceable if there exists a hamiltonian path between any two vertices that are in different partite sets. A hamiltonian laceable graph G is said to be hyper-hamiltonian laceable if, for any vertex v of G, there exists a hamiltonian path of $G-\{v\}$ joining any two vertices that are located in the same partite set different from that of v. In this paper, we further improve the hyper-hamiltonian laceability of hypercubes by showing that, for any two vertices $\mathbf{x}$, $\mathbf{y}$ from one partite set of Q_n, $n \geq 4$, and any vertex $\mathbf{w}$ from the other partite set, there exists a hamiltonian path H of $Q_n - \{\mathbf{w}\}$ joining $\mathbf{x}$ to $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = l$ for any vertex $\mathbf{z} \in V(Q_n) - \{\mathbf{x}, \mathbf{y}, \mathbf{w}\}$ and for every integer l satisfying both $d_{Q_n}(\mathbf{x}, \mathbf{z}) \leq l \leq 2^n - 2 - d_{Q_n}(\mathbf{z}, \mathbf{y})$ and $2|(l - d_{Q_n}(\mathbf{x}, \mathbf{z}))$. As a consequence, many attractive properties of hypercubes follow directly from our result.

***Keywords*—** Path embedding, Hamiltonian laceable, Hyper-hamiltonian laceable, Interconnection network, Hypercube

I. Introduction

A multiprocessor/multicomputer interconnection network is usually modeled as a graph [2], [4], [8], [14], [16], [19], [20], in which vertices correspond to processors/computers, and edges correspond to connections or communication links. Throughout this paper, a network is represented as a loopless undirected graph. For graph definitions and notations, we follow the ones given by Hsu and Lin [4]. A graph G is a two-tuple (V, E), where V is a nonempty set, and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the vertex set and E is the edge set. For convenience, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. A graph G is bipartite if its vertex set $V(G)$ is the union of two disjoint subsets, denoted by $V_0(G)$ and $V_1(G)$, such that every edge joins a vertex of $V_0(G)$ to a vertex of $V_1(G)$. Two vertices, u and v, of a graph G are adjacent if $(u, v) \in E(G)$. A *path* P of length k from vertex x to vertex y in a graph G is a sequence of distinct vertices $\langle v_1, v_2, \ldots, v_{k+1}\rangle$ such that $v_1 = x$, $v_{k+1} = y$, and $(v_i, v_{i+1}) \in E(G)$ for every $1 \leq i \leq k$ if $k \geq 1$. Moreover, a path of length zero, consisting of a single vertex x, is denoted by $\langle x\rangle$. We also write P as $\langle x, P, y\rangle$ to emphasize its beginning and ending vertices. For convenience, we write P as $\langle v_1, \ldots, v_i, Q, v_j, \ldots, v_{k+1}\rangle$, where $Q = \langle v_i, \ldots, v_j\rangle$. In particular, let $P^{-1} = \langle v_{k+1}, v_k, \ldots, v_1\rangle$ denote the reverse of P. A *cycle* is a path with at least three vertices such that the last vertex is adjacent to the first one. For clarity, a cycle of length k is represented by $\langle v_1, v_2, \ldots, v_k, v_1\rangle$. The length of a path P, denoted by $\ell(P)$, is the number of edges in P. The *distance* between two distinct vertices u and v in graph G, denoted by $d_G(u, v)$, is the length of the shortest path between u and v. A *hamiltonian cycle* (or *hamiltonian path*) of a graph G is a cycle (or path) that spans G. A bipartite graph is *hamiltonian laceable* [15] if there exists a hamiltonian path between any two vertices that are in different partite sets. A hamiltonian laceable graph G is *hyper-hamiltonian laceable* [9] if, for $i \in \{0, 1\}$ and for any vertex $v \in V_i(G)$, there exists a hamiltonian path of $G - \{v\}$ between any two vertices of $V_{1-i}(G)$.

The hypercube is one of the most popular interconnection networks for parallel computer/communication system [4], [8], [19]. This is partly due to its attractive properties such as regularity, recursive structure, vertex and edge symmetry, maximum connectivity, as well as effective routing and broadcasting algorithm. The definition of hypercubes is presented as follows. For clarity, we use boldface letters to denote n-bit binary strings. Let $\mathbf{u} = b_{n-1} \ldots b_i \ldots b_0$ be an n-bit binary string. For any i, $0 \leq i \leq n-1$, we use $(\mathbf{u})^i$ to denote the binary string $b_{n-1} \ldots \bar{b}_i \ldots b_0$. Moreover, we use $(\mathbf{u})_i$ to denote the bit b_i of $\mathbf{u}$. The *Hamming weight* of $\mathbf{u}$, denoted by $w_H(\mathbf{u})$, is defined as $|\{0 \leq j \leq n-1 \mid (\mathbf{u})_j = 1\}|$. The *$n$-dimensional hypercube* (or *n-cube* for short) Q_n consists of 2^n vertices and $n2^{n-1}$ edges. Each vertex corresponds to an n-bit binary string. Two vertices $\mathbf{u}$ and $\mathbf{v}$ are adjacent if and only if $\mathbf{v} = (\mathbf{u})^i$ for some i, and we call the edge $(\mathbf{u}, (\mathbf{u})^i)$ i-dimensional. The *Hamming distance* between $\mathbf{u}$ and $\mathbf{v}$, denoted by $h(\mathbf{u}, \mathbf{v})$, is defined to be $|\{0 \leq j \leq n-1 \mid (\mathbf{u})_j \neq (\mathbf{v})_j\}|$. Hence, two vertices $\mathbf{u}$ and $\mathbf{v}$ are adjacent if and only if $h(\mathbf{u}, \mathbf{v}) = 1$. Clearly, $d_{Q_n}(\mathbf{u}, \mathbf{v})$ equals $h(\mathbf{u}, \mathbf{v})$, and Q_n is a bipartite graph with partite sets $V_0(Q_n) = \{\mathbf{v} \in V(Q_n) \mid w_H(\mathbf{v})$ is even$\}$ and $V_1(Q_n) = \{\mathbf{v} \in V(Q_n) \mid w_H(\mathbf{v})$ is odd$\}$. Moreover, Q_n is vertex-symmetric and edge-symmetric [8].

The problem of embedding paths into hypercube is widely addressed by many researchers [1], [3], [5], [6], [10], [12], [13], [18]. In particular, Li et al. [10] proved that between any two different vertices $\mathbf{x}$ and $\mathbf{z}$ of Q_n there exists a path

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$P_l(\mathbf{x},\mathbf{z})$ of length l for any l with $h(\mathbf{x},\mathbf{z}) \leq l \leq 2^n - 1$ and $2|(l - h(\mathbf{x},\mathbf{z}))$. It is intriguing to consider whether such path $P_l(\mathbf{x},\mathbf{z})$ can be further extended by including the remaining vertices not in $P_l(\mathbf{x},\mathbf{z})$ to form a hamiltonian path between $\mathbf{x}$ and some other vertex $\mathbf{y}$. For this motivation, Lee et al. [7] showed that, for any two vertices $\mathbf{x}$ and $\mathbf{y}$ from different partite set of Q_n and for any vertex $\mathbf{z} \in V(Q_n) - \{\mathbf{x},\mathbf{y}\}$, there exists a hamiltonian path R from $\mathbf{x}$ to $\mathbf{y}$ such that $d_R(\mathbf{x},\mathbf{z}) = l$ for any integer l satisfying $h(\mathbf{x},\mathbf{z}) \leq l \leq 2^n - 1 - h(\mathbf{z},\mathbf{y})$ and $2|(l - h(\mathbf{x},\mathbf{z}))$. In this paper, we further improve such result by proving that, for any two vertices $\mathbf{x}$, $\mathbf{y}$ from one partite set of Q_n and any vertex $\mathbf{w}$ from the other partite set, there exists a hamiltonian path H between $\mathbf{x}$ and $\mathbf{y}$ in $Q_n - \{\mathbf{w}\}$ such that $d_H(\mathbf{x},\mathbf{z}) = l$ for any vertex $\mathbf{z} \in V(Q_n) - \{\mathbf{x},\mathbf{y},\mathbf{w}\}$ and for every integer l satisfying both $d_{Q_n}(\mathbf{x},\mathbf{z}) \leq l \leq 2^n - 2 - d_{Q_n}(\mathbf{z},\mathbf{y})$ and $2|(l - d_{Q_n}(\mathbf{x},\mathbf{z}))$. According to the proposed improvement, many attractive properties of hypercubes follow directly from our result.

II. Preliminaries

Lemma 1-4 and Theorem 5 were proved in [7], [9], [15], [17]. They will be used to prove our main result in the next section.

Lemma 1: [15] For any positive integer n, Q_n is hamiltonian laceable.

Lemma 2: [9] For any positive integer n, Q_n is hyper-hamiltonian laceable.

Lemma 3: [17] Suppose that $n \geq 4$. If $\mathbf{x}$ and $\mathbf{y}$ are any two vertices from different sets of Q_n, then $Q_n - \{\mathbf{x},\mathbf{y}\}$ is hamiltonian laceable.

Two paths P_1 and P_2 are *disjoint* if $V(P_1) \cap V(P_2) = \emptyset$. The following lemma is proved by Lee et al. [7].

Lemma 4: [7] Assume that $n \geq 4$. Let $\mathbf{u}$, $\mathbf{x}$ be any pair of distinct vertices in $V_0(Q_n)$. Moreover, let $\mathbf{v}$, $\mathbf{y}$ be any pair of distinct vertices in $V_1(Q_n)$. Suppose that l_1 and l_2 are two arbitrary odd integers satisfying $l_1 \geq h(\mathbf{u},\mathbf{v})$, $l_2 \geq h(\mathbf{x},\mathbf{y})$, and $l_1 + l_2 = 2^n - 2$. Then there exist two disjoint paths P_1 and P_2 such that the following two conditions are satisfied: (1) P_1 joins $\mathbf{u}$ to $\mathbf{v}$ with $\ell(P_1) = l_1$, and (2) P_2 joins $\mathbf{x}$ to $\mathbf{y}$ with $\ell(P_2) = l_2$.

Now, we make some remarks to illustrate that some interesting properties of hypercubes are consequences of Lemma 4.

Remark 5: The hamiltonian laceable property of hypercubes, proved in [15], states that there exists a hamiltonian path of Q_n joining any vertex $\mathbf{u} \in V_0(Q_n)$ to any vertex $\mathbf{y} \in V_1(Q_n)$. Now, we prove that Q_n is hamiltonian laceable by Lemma 4. Obviously, Q_n is hamiltonian laceable for $n = 1, 2, 3$. Since $n \geq 4$, we can choose a pair of adjacent vertices $\mathbf{v}$ and $\mathbf{x}$ such that $\mathbf{v} \in V_1(Q_n)$ with $\mathbf{v} \neq \mathbf{y}$ and $\mathbf{x} \in V_0(Q_n)$ with $\mathbf{x} \neq \mathbf{u}$. By Lemma 4, there are two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{u}$ to $\mathbf{v}$, (2) P_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$, and (3) $P_1 \cup P_2$ spans Q_n. Obviously, $\langle \mathbf{u}, P_1, \mathbf{v}, \mathbf{x}, P_2, \mathbf{y} \rangle$ forms a hamiltonian path joining $\mathbf{u}$ to $\mathbf{y}$. Thus, Q_n is hamiltonian laceable.

Remark 6: The bipanconnected property of Q_n, proved in [10], stated that between any two different vertices $\mathbf{x}$ and $\mathbf{y}$ of Q_n there exists a path $P_l(\mathbf{x},\mathbf{y})$ of length l for any l with $h(\mathbf{x},\mathbf{y}) \leq l \leq 2^n - 1$ and $2 \mid (l - h(\mathbf{x},\mathbf{y}))$. Now, we prove that Q_n is bipanconnected by Lemma 4. Obviously, Q_n is bipanconnected for $n = 1, 2, 3$. Now, we consider $n \geq 4$. Without loss of generality, we assume that $\mathbf{x} \in V_0(Q_n)$.

Suppose that $\mathbf{y} \in V_1(Q_n)$. Thus, $h(\mathbf{x},\mathbf{y})$ is odd. Let l be any odd integer with $h(\mathbf{x},\mathbf{y}) \leq l \leq 2^n - 1$. Suppose that $l = 2^n - 1$. By Remark 5, Q_n is hamiltonian laceable. Obviously, the hamiltonian path of Q_n joining $\mathbf{x}$ and $\mathbf{y}$ is of length $2^n - 1$. Suppose that $l < 2^n - 1$. Since $n \geq 4$, we can choose a pair of adjacent vertices $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u} \in V_0(Q_n)$ with $\mathbf{u} \neq \mathbf{x}$ and $\mathbf{v} \in V_0(Q_n)$ with $\mathbf{v} \neq \mathbf{y}$. Obviously, $h(\mathbf{u},\mathbf{v}) = 1$. By Lemma 4, there exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{u}$ to $\mathbf{v}$ with $l(P_1) = 2^n - 2 - l$, (2) P_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(P_2) = l$, and (3) $P_1 \cup P_2$ spans Q_n. Obviously, P_2 is a path of length l joining $\mathbf{x}$ to $\mathbf{y}$.

Suppose that $\mathbf{y} \in V_0(Q_n)$. Thus, $h(\mathbf{x},\mathbf{y})$ is even. Let l be any even integer with $h(\mathbf{x},\mathbf{y}) \leq l < 2^n - 1$. Since $n \geq 4$, we can choose two different neighbors $\mathbf{u}$ and $\mathbf{v}$ of $\mathbf{y}$ such that $h(\mathbf{x},\mathbf{u}) = h(\mathbf{x},\mathbf{y}) - 1$. By Lemma 4, there exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{x}$ to $\mathbf{u}$ with $l(P_1) = l - 1$, (2) P_2 is a path joining $\mathbf{y}$ to $\mathbf{v}$ with $l(P_2) = 2^n - l - 1$, and (3) $P_1 \cup P_2$ spans Q_n. Obviously, $\langle \mathbf{x}, P_1, \mathbf{u}, \mathbf{y} \rangle$ is a path of length l joining $\mathbf{x}$ to $\mathbf{y}$.

Thus, Q_n is bipanconnected.

Remark 7: The edge-bipancyclic property property of Q_n, proved in [10], stated that for any edge $e = (\mathbf{x},\mathbf{y})$ and for any even integer with $4 \leq l \leq 2^n$ there exists a cycle of length l containing the edge e if $n \geq 2$. Again, we we prove that Q_n is edge-bipancyclic by Lemma 4. Obviously, Q_n is edge-bipancyclic for $n = 2, 3$. Thus, we consider $n \geq 4$. Suppose that $l = 2^n$. By Remark 5, there exists a hamiltonian path P joining $\mathbf{x}$ to $\mathbf{y}$. Obviously, $\langle \mathbf{x}, P, \mathbf{y}, \mathbf{x} \rangle$ forms a hamiltonian cycle of length 2^n containing the edge e. Suppose that $l < 2^n$. Since $n \geq 4$, we can choose a pair of adjacent vertices $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u} \in V_0(Q_n)$. By Lemma 4, there exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{u}$ to $\mathbf{v}$ with $l(P_1) = 2^n - l - 1$, (2) P_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(P_2) = l - 1$, and (3) $P_1 \cup P_2$ spans Q_n. Obviously, $\langle \mathbf{x}, P_2, \mathbf{y}, \mathbf{x} \rangle$ is a cycle of length l containing the edge e. Thus, Q_n is edge-bipancyclic for $n \geq 2$.

Now, we start to prove Lemma 4.

Proof: By brute force, we can check the theorem holds for $n = 4$. Assume the theorem holds for any Q_k with $4 \leq k < n$. Without loss of generality, we can assume that $l_1 \geq l_2$. Thus, $l_2 \leq 2^{n-1} - 1$. Since Q_n is vertex-symmetric and edge-symmetric, we can assume that $\mathbf{u} \in V_0(Q^0_{n-1})$ and $\mathbf{x} \in V_0(Q^1_{n-1})$. We have the following cases.

Case 1: $\mathbf{v} \in V_1(Q^0_{n-1})$ and $\mathbf{y} \in V_1(Q^1_{n-1})$.

Suppose that $l_2 < 2^{n-1} - 1$. By Remark 5, there exists a hamiltonian path R of Q^0_{n-1} joining $\mathbf{u}$ and $\mathbf{v}$. Since the length of R is $2^{n-1} - 1$, we can write R as $\langle \mathbf{u}, R_1, \mathbf{p}, \mathbf{q}, R_2, \mathbf{v} \rangle$ for some vertex $\mathbf{p} \in V_1(Q_n)$ with $\mathbf{p}^n \neq \mathbf{x}$ and some vertex $\mathbf{q} \in V_0(Q_n)$ with $\mathbf{q}^n \neq \mathbf{y}$. Obviously, $h(\mathbf{p}^n, \mathbf{q}^n) = 1$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{p}^n$ to $\mathbf{q}^n$ with $l(S_1) = l_1 - 2^{n-1}$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(S_2) = l_2$, and (3) $S_1 \cup S_2$ spans Q^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{p}, \mathbf{p}^n, S_1, \mathbf{q}^n, \mathbf{q}, R_2, \mathbf{v} \rangle$ and set P_2 as S_2. Obviously, P_1 and P_2 are the required paths. See Figure 4(a) for illustration.

Suppose that $l_2 = 2^{n-1} - 1$. By Remark 5, there exists a hamiltonian path P_1 of Q^0_{n-1} joining $\mathbf{u}$ and $\mathbf{v}$ and there exists a hamiltonian path P_2 of Q^1_{n-1} joining $\mathbf{x}$ and $\mathbf{y}$. Obviously, P_1 and P_2 are the required paths. See Figure 4(b) for illustration.

Case 2: $\{\mathbf{v}, \mathbf{y}\} \subset V_1(Q^1_{n-1})$.

Suppose that $l_2 < 2^{n-1} - 1$. We choose a neighbor $\mathbf{p}$ of $\mathbf{v}$ such that $\mathbf{p} \neq \mathbf{x}$. Obviously, $\mathbf{p} \in V_0(Q_n)$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{p}$ to $\mathbf{v}$ with $l(S_1) = l_1 - 2^{n-1}$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(S_2) = l_2$, and (3) $S_1 \cup S_2$ spans Q^1_{n-1}. By Remark 5, there exists a hamiltonian path R of Q^0_{n-1} joining $\mathbf{u}$ and $\mathbf{p}^n$. We set P_1 as $\langle \mathbf{u}, R, \mathbf{p}^n, \mathbf{p}, S_1, \mathbf{v}\rangle$ and we set P_2 as S_2. Obviously, P_1 and P_2 are the required paths. See Figure 4(c) for illustration.

Suppose that $l_2 = 2^{n-1} - 1$. Again, we choose a neighbor $\mathbf{p}$ of $\mathbf{v}$ such that $\mathbf{p} \neq \mathbf{x}$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{p}$ to $\mathbf{v}$ with $l(S_1) = 1$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(S_2) = 2^{n-1} - 3$, and (3) $S_1 \cup S_2$ spans Q^1_{n-1}. Obviously, we can write S_2 as $\langle \mathbf{x}, S^1_2, \mathbf{r}, \mathbf{s}, S^2_2, \mathbf{y}\rangle$ for some vertex $\mathbf{r} \in V_1(Q_n)$ with $\mathbf{r}^n \neq \mathbf{u}$. Again by induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{p}^n$ with $l(R_1) = 2^{n-1} - 3$, (2) R_2 is a path joining $\mathbf{r}^n$ to $\mathbf{s}^n$ with $l(R_2) = 1$, and (3) $R_1 \cup R_2$ spans Q^0_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{p}^n, \mathbf{p}, \mathbf{v}\rangle$ and set P_2 as $\langle \mathbf{x}, S^1_2, \mathbf{r}, \mathbf{r}^n, \mathbf{s}^n, \mathbf{s}, S^2_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths. See Figure 4(d) for illustration.

Case 3: $\mathbf{y} \in V_1(Q^0_{n-1})$ and $\mathbf{v} \in V_1(Q^1_{n-1})$.

Suppose that $l_2 = 1$. Obviously, $\mathbf{x} = \mathbf{y}^n$. Let $\mathbf{p}$ be a neighbor of $\mathbf{y}$ in Q^0_{n-1} such that $\mathbf{y}^n \neq \mathbf{v}$ and let $\mathbf{q}$ be a neighbor of $\mathbf{p}$ in Q^0_{n-1} such that $\mathbf{p} \neq \mathbf{y}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{q}$ and $l(R_1) = 2^{n-1}-3$, (2) R_2 is a path joining $\mathbf{p}$ to $\mathbf{y}$ and $l(R_2) = 1$, and (3) $R_1 \cup R_2$ spans Q^0_{n-1}. Obviously, $\mathbf{p}^n \in V_1(Q_n)$ and $\mathbf{q}^n \in V_0(Q_n)$. Again by induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{q}^n$ to $\mathbf{v}$ with $l(S_1) = 2^{n-1} - 3$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{p}^n$ with $l(S_2) = 1$, and (3) $S_1 \cup S_2$ spans Q^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{q}, \mathbf{p}, \mathbf{p}^n, \mathbf{q}^n, S_1, \mathbf{v}\rangle$ and set P_2 as $\langle \mathbf{x}, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths. See Figure 4(e) for illustration.

Suppose that $l_2 \geq 3$. We set $\mathbf{p}$ be a neighbor in Q^0_{n-1} of $\mathbf{y}$ with $\mathbf{p} \neq \mathbf{u}$ if $h(\mathbf{x}, \mathbf{y}) = 1$ and set $\mathbf{p}$ be a neighbor of $\mathbf{y}$ in Q^0_{n-1} with $\mathbf{p} \neq \mathbf{u}$ and $h(\mathbf{p}, \mathbf{y}) = h(\mathbf{x}, \mathbf{y}) - 1$ if $h(\mathbf{x}, \mathbf{y}) \geq 3$. Let $\mathbf{q}$ be a neighbor $\mathbf{v}^n$ in Q^0_{n-1} such that $\mathbf{q} \neq \mathbf{y}$ and $\mathbf{q}^n \neq \mathbf{x}$. Thus, $h(\mathbf{q}^n, \mathbf{v}) = 1$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{p}$ with $l(R_1) = 2^{n-1} - 3$, (2) R_2 is a path joining $\mathbf{q}$ to $\mathbf{y}$ with $l(R_2) = 1$, and (3) $R_1 \cup R_2$ spans Q^0_{n-1}. Again by induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{q}^n$ to $\mathbf{v}$ with $l(S_1) = l_1 - 2^{n-1} + 2$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{p}^n$ with $l(S_2) = 1_2 - 2$, and (3) $S_1 \cup S_2$ spans Q^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{q}, \mathbf{q}^n, S_1, \mathbf{v}\rangle$ and set P_2 as $\langle \mathbf{x}, S_2, \mathbf{p}^n, \mathbf{p}, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths. See Figure 4(f) for illustration.

Case 4: $\{\mathbf{v}, \mathbf{y}\} \subset V_1(Q^0_{n-1})$.

Suppose that $l_2 = 1$. Obviously, $\mathbf{y} = \mathbf{x}^n$. By Remark 5, there exist a hamiltonian path R of Q^0_{n-1} joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, R can be written as $\langle \mathbf{u}, R_1, \mathbf{p}, \mathbf{y}, \mathbf{q}, R_2, \mathbf{v}\rangle$. Note that $\mathbf{u} = \mathbf{p}$ if $l(R_1) = 0$. Obviously, $\mathbf{p}$ and $\mathbf{q}$ are in $V_0(Q_n)$. Thus, $\mathbf{p}^n$ and $\mathbf{q}^n$ are in $V_1(Q_n)$. Let $\mathbf{r}$ be a neighbor of $\mathbf{q}^n$ in Q^1_{n-1} such that $\mathbf{r} \neq \mathbf{x}$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{p}^n$ to $\mathbf{r}$ with $l(S_1) = 2^{n-1} - 3$, (2) S_2 is a path joining $\mathbf{q}^n$ to $\mathbf{x}$ with $l(S_2) = 1$, and (3) $S_1 \cup S_2$ spans Q^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{p}, \mathbf{p}^n, S_1, \mathbf{r}, \mathbf{q}^n, \mathbf{q}, R_2, \mathbf{v}\rangle$ and set P_2 as $\langle \mathbf{x}, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths. See Figure 4(g) for illustration.

Suppose that $l_2 \geq 3$. We set $\mathbf{p}$ be a neighbor of $\mathbf{y}$ in Q^0_{n-1} with $\mathbf{p} \neq \mathbf{u}$ if $h(\mathbf{x}, \mathbf{y}) = 1$ and set $\mathbf{p}$ be a neighbor of $\mathbf{y}$ in Q^0_{n-1} with $\mathbf{p} \neq \mathbf{u}$ and $h(\mathbf{p}, \mathbf{y}) = h(\mathbf{x}, \mathbf{y}) - 1$ if $h(\mathbf{x}, \mathbf{y}) \geq 3$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{v}$ with $l(R_1) = 2^{n-1} - 3$, (2) R_2 is a path joining $\mathbf{p}$ to $\mathbf{y}$ with $l(R_2) = 1$, and (3) $R_1 \cup R_2$ spans Q^0_{n-1}. Obviously, we can write R_1 as $\langle \mathbf{u}, R^1_1, \mathbf{s}, \mathbf{t}, R^2_1, \mathbf{v}\rangle$ for some vertex $\mathbf{s} \in V_1(Q_n)$ such that $\mathbf{s}^n \neq \mathbf{x}$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{s}^n$ to $\mathbf{t}^n$ with $l(S_1) = l_1 - 2^{n-1} - 2$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{p}^n$ with $l(S_2) = l_2 - 2$, and (3) $S_1 \cup S_2$ spans Q^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R^1_1, \mathbf{s}, \mathbf{s}^n, S_1, \mathbf{t}^n, \mathbf{t}, R^2_1, \mathbf{v}\rangle$ and set P_2 as $\langle \mathbf{x}, S_2, \mathbf{p}^n, \mathbf{p}, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths. See Figure 4(h) for illustration. ■

By Lemma 4, the next result can be derived.

Theorem 8: [7] Assume that $n \geq 2$. Let $\mathbf{x}$ and $\mathbf{y}$ be any two vertices from different partite sets of Q_n, and let $\mathbf{z}$ be any vertex of $Q_n - \{\mathbf{x}, \mathbf{y}\}$. Then there exists a hamiltonian path H of Q_n joining $\mathbf{x}$ to $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = l$ for any integer l satisfying both $h(\mathbf{x}, \mathbf{z}) \leq l \leq 2^n - 1 - h(\mathbf{z}, \mathbf{y})$ and $2|(l - h(\mathbf{x}, \mathbf{z}))$.

Proof: By brute force, we can check the theorem holds for $n = 2, 3$. Now, we consider $n \geq 4$. Without loss of generality, we assume that $\mathbf{x} \in V_0(Q_n)$ and $\mathbf{z} \in V_1(Q_n)$.

Suppose that $\mathbf{y} \in V_1(Q_n)$. Obviously, $h(\mathbf{y}, \mathbf{z}) \geq 2$. There exists a neighbor $\mathbf{w}$ of $\mathbf{y}$ such that $\mathbf{w} \neq \mathbf{x}$ and $h(\mathbf{w}, \mathbf{z}) = h(\mathbf{y}, \mathbf{z}) - 1$. Obviously, $\mathbf{w} \in V_0(Q_n)$. By Lemma 4, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(R_1) = l$, (2) R_2 is a path joining $\mathbf{w}$ to $\mathbf{z}$ with $l(R_2) = 2^n - l - 2$, and (3) $R_1 \cup R_2$ spans Q_n. We set R as $\langle \mathbf{x}, R_1, \mathbf{y}, \mathbf{w}, R_2, \mathbf{z}\rangle$. Obviously, R is the required hamiltonian path.

Suppose that $\mathbf{y} \in V_0(Q_n)$. Obviously, $h(\mathbf{x}, \mathbf{y}) \geq 2$. There exists a neighbor $\mathbf{w}$ of $\mathbf{y}$ such that $\mathbf{w} \neq \mathbf{z}$ and $h(\mathbf{w}, \mathbf{x}) = h(\mathbf{y}, \mathbf{x}) - 1$. Obviously, $\mathbf{w} \in V_1(Q_n)$. By Lemma 4, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{x}$ to $\mathbf{w}$ with $l(R_1) = l - 1$, (2) R_2 is a path joining $\mathbf{y}$ to $\mathbf{z}$ with $l(R_2) = 2^n - l - 1$, and (3) $R_1 \cup R_2$ spans Q_n. We set R as $\langle \mathbf{x}, R_1, \mathbf{w}, \mathbf{y}, R_2, \mathbf{z}\rangle$. Obviously, R is the required hamiltonian path. ■

III. Hyper-hamiltonian Laceability

For $0 \leq j \leq n-1$ and $i \in \{0, 1\}$, let $Q^{j,i}_n$ be a subgraph of Q_n induced by $\{u \in V(Q_n) \mid (u)_j = i\}$. Obviously, $Q^{j,i}_n$ is

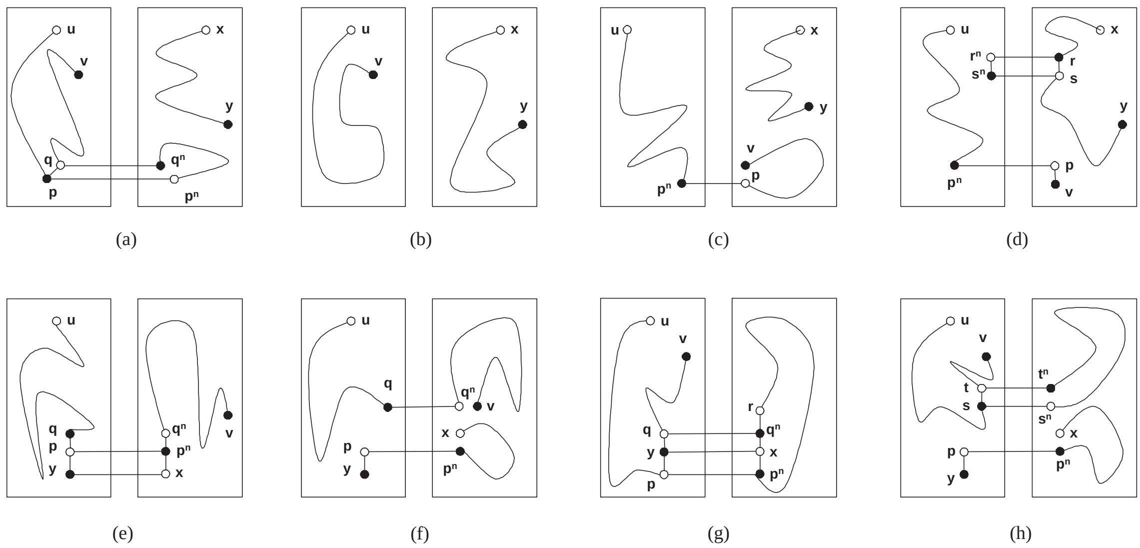

Fig. 1. Illustration for Lemma 4.

isomorphic to Q_{n-1}. Then we can prove our main result by induction.

Theorem 9: Suppose that $n \geq 4$. Let $\mathbf{x}$, $\mathbf{y}$ be any pair of distinct vertices in $V_1(Q_n)$ and $\mathbf{w}$ be any vertex of $V_0(Q_n)$. Moreover, let $\mathbf{z} \in V(Q_n) - \{\mathbf{x}, \mathbf{y}, \mathbf{w}\}$. Then $Q_n - \{\mathbf{w}\}$ contains a hamiltonian path H between $\mathbf{x}$ and $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = l$ for every integer l satisfying both $d_{Q_n}(\mathbf{x}, \mathbf{z}) \leq l \leq 2^n - 2 - d_{Q_n}(\mathbf{z}, \mathbf{y})$ and $2|(l - d_{Q_n}(\mathbf{x}, \mathbf{z}))$.

Proof: We prove this theorem by induction on n. As the induction basis, we check, by computer program, that this result holds for $n = 4$. Assume that the result is true for any integer k with $4 \leq k < n$. Since Q_n is vertex-symmetric and edge-symmetric, we assume that $\mathbf{w} = 0^n$. If $\mathbf{z} \in V_0(Q_n)$, then let j be an integer of $\{0, 1, \ldots, n-1\}$ such that $(\mathbf{x})_j \neq (\mathbf{y})_j$ and $(\mathbf{x})^j \neq \mathbf{z}$. Otherwise, let j be an integer of $\{0, 1, \ldots, n-1\}$ such that $(\mathbf{x})_j \neq (\mathbf{y})_j$ and $(\mathbf{w})^j \neq \mathbf{z}$. Then we can partition Q_n along dimension j into $Q_n^{j,0}$ and $Q_n^{j,1}$. Without loss of generality, we assume that $\mathbf{x}$ is in $Q_n^{j,0}$, and $\mathbf{y}$ is in $Q_n^{j,1}$. To construct a hamiltonian path H of $Q_n - \{\mathbf{w}\}$ joining $\mathbf{x}$ to $\mathbf{y}$ with $d_H(\mathbf{x}, \mathbf{z}) = l$, the following cases are distinguished.

Case 1: Suppose that $\mathbf{z} \in V_0(Q_n^{j,0})$. We further consider the following subcases.

Subcase 1.1: Suppose that $l \leq 2^{n-1} - 3$. Let $\mathbf{v}$ be a vertex of $Q_n^{j,0}$ adjacent to $\mathbf{z}$. By inductive hypothesis, $Q_n^{j,0} - \{\mathbf{w}\}$ has a hamiltonian path L between $\mathbf{x}$ and $\mathbf{v}$ such that $d_L(\mathbf{x}, \mathbf{z}) = l$ for every integer l satisfying both $d_{Q_n}(\mathbf{x}, \mathbf{z}) \leq l \leq 2^{n-1} - 2 - d_{Q_n}(\mathbf{z}, \mathbf{v}) = 2^{n-1} - 3$ and $2|(l - d_{Q_n}(\mathbf{x}, \mathbf{z}))$. Thus we can write L as $\langle \mathbf{x}, L_1, \mathbf{z}, L_2, \mathbf{v} \rangle$. By Lemma 1, $Q_n^{j,1}$ is hamiltonian laceable. Since $(\mathbf{v})^j$ and $\mathbf{y}$ are in the different partite sets, $Q_n^{j,1}$ has a hamiltonian path R between $(\mathbf{v})^j$ and $\mathbf{y}$. Let $H = \langle \mathbf{x}, L_1, \mathbf{z}, L_2, \mathbf{v}, (\mathbf{v})^j, R, \mathbf{y} \rangle$. As a result, H is a hamiltonian path of $Q_n - \{\mathbf{w}\}$ joining $\mathbf{x}$ to $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = l$ for any odd integer $l \leq 2^{n-1} - 3$. See Figure 2(a) for illustration.

Subcase 1.2: Suppose that $l \geq 2^{n-1} - 1$. Let $k \in \{0, 1, \ldots, n-1\} - \{j\}$ such that $(\mathbf{z})_k \neq (\mathbf{y})_k$. Then we set $\mathbf{v}$ to be $(\mathbf{z})^k$. Trivially we have $d_{Q_n}(\mathbf{v}, \mathbf{z}) = 1$.

Firstly, we consider that $(\mathbf{z})^j \neq \mathbf{y}$. By inductive hypothesis, $Q_n^{j,0} - \{\mathbf{w}\}$ has a hamiltonian path L between $\mathbf{x}$ and $\mathbf{v}$ such that $d_L(\mathbf{v}, \mathbf{z}) = 1$. Accordingly, we can write L as $\langle \mathbf{x}, L_1, \mathbf{z}, \mathbf{v} \rangle$. Since $\ell(L_1) = 2^{n-1} - 3 \geq 5$ for $n \geq 4$, we write L_1 as $\langle \mathbf{x}, L_1', \mathbf{u}, \mathbf{z} \rangle$, where $\mathbf{u}$ is some vertex of $Q_n^{j,0}$ adjacent to $\mathbf{z}$. Hence path L can be represented as $\langle \mathbf{x}, L_1', \mathbf{u}, \mathbf{z}, \mathbf{v} \rangle$. By Lemma 4, there exist two disjoint paths R_1 and R_2 in $Q_n^{j,1}$ such that R_1 joins $(\mathbf{u})^j$ to $(\mathbf{z})^j$ with $\ell(R_1) = l - 2^{n-1} + 2$, and R_2 joins $(\mathbf{v})^j$ to $\mathbf{y}$ with $\ell(R_2) = 2^n - l - 4$. Let $H = \langle \mathbf{x}, L_1', \mathbf{u}, (\mathbf{u})^j, R_1, (\mathbf{z})^j, \mathbf{z}, \mathbf{v}, (\mathbf{v})^j, R_2, \mathbf{y} \rangle$. As a result, H turns out to be a hamiltonian path of $Q_n - \{\mathbf{w}\}$ between $\mathbf{x}$ and $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = l$ for any odd integer $l \geq 2^{n-1} - 1$ if $(\mathbf{z})^j \neq \mathbf{y}$. See Figure 2(b) for illustration.

Secondly, we consider that $(\mathbf{z})^j = \mathbf{y}$. Again, the inductive hypothesis ensures that $Q_n^{j,0} - \{\mathbf{w}\}$ has a hamiltonian path L between $\mathbf{x}$ and $\mathbf{v}$ such that $d_L(\mathbf{v}, \mathbf{z}) = 1$. Hence we can write L as $\langle \mathbf{x}, L_1, \mathbf{z}, \mathbf{v} \rangle$. Since $\ell(L_1) = 2^{n-1} - 3 \geq 5$ for $n \geq 4$, we write L_1 as $\langle \mathbf{x}, \mathbf{r}, L_1', \mathbf{z} \rangle$, where $\mathbf{r}$ is some vertex of $Q_n^{j,0}$ adjacent to $\mathbf{x}$. Accordingly, path L can be represented as $\langle \mathbf{x}, \mathbf{r}, L_1', \mathbf{z}, \mathbf{v} \rangle$. By Lemma 4, there exist two disjoint paths R_1 and R_2 in $Q_n^{j,1}$ such that R_1 is a path of length $l - 2^{n-1} + 2$ joining $(\mathbf{x})^j$ to $(\mathbf{r})^j$, and R_2 is a path of length $2^n - l - 4$ joining $(\mathbf{v})^j$ to $\mathbf{y}$. Let $H = \langle \mathbf{x}, (\mathbf{x})^j, R_1, (\mathbf{r})^j, \mathbf{r}, L_1', \mathbf{z}, \mathbf{v}, (\mathbf{v})^j, R_2, \mathbf{y} \rangle$. As a consequence, H forms a hamiltonian path of $Q_n - \{\mathbf{w}\}$ between $\mathbf{x}$ and $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = l$ if $(\mathbf{z})^j = \mathbf{y}$. See Figure 2(c) for illustration.

Case 2: Suppose that $\mathbf{z} \in V_1(Q_n^{j,0})$. Similar to the case described earlier, we consider the following subcases.

Subcase 2.1: Suppose that $l \leq 2^{n-1} - 4$. Let $\mathbf{v} \in V_1(Q_n^{j,0})$ such that $d_{Q_n}(\mathbf{v}, \mathbf{z}) = 2$. By inductive hypothesis, $Q_n^{j,0} - \{\mathbf{w}\}$ has a hamiltonian path L between $\mathbf{x}$ and $\mathbf{v}$ such that $d_L(\mathbf{x}, \mathbf{z}) = l$ for every integer l satisfying both $d_{Q_n}(\mathbf{x}, \mathbf{z}) \leq l \leq 2^{n-1} - 2 - d_{Q_n}(\mathbf{z}, \mathbf{v}) = 2^{n-1} - 4$ and $2|(l - d_{Q_n}(\mathbf{x}, \mathbf{z}))$. For clarity, we write L as $\langle \mathbf{x}, L_1, \mathbf{z}, L_2, \mathbf{v} \rangle$. Since $(\mathbf{v})^j$ and $\mathbf{y}$ are in the different partite sets of Q_n, Lemma 1 ensures that $Q_n^{j,1}$ has a hamiltonian path R between $(\mathbf{v})^j$ and $\mathbf{y}$. Let $H = \langle \mathbf{x}, L_1, \mathbf{z}, L_2, \mathbf{v}, (\mathbf{v})^j, R, \mathbf{y} \rangle$. Consequently, path H turns

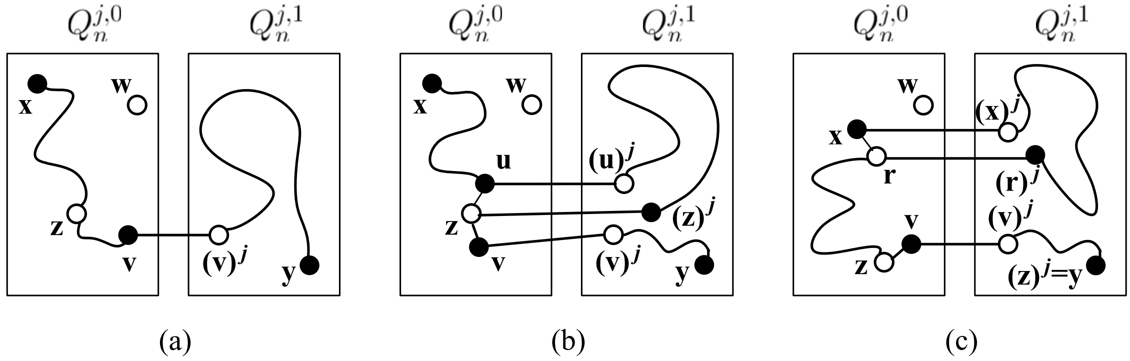

Fig. 2. $\mathbf{z} \in V_0(Q_n^{j,0})$. (a) $l \leq 2^{n-1} - 3$. (b) $l \geq 2^{n-1} - 1$ and $(\mathbf{z})^j \neq \mathbf{y}$. (c) $l \geq 2^{n-1} - 1$ and $(\mathbf{z})^j = \mathbf{y}$.

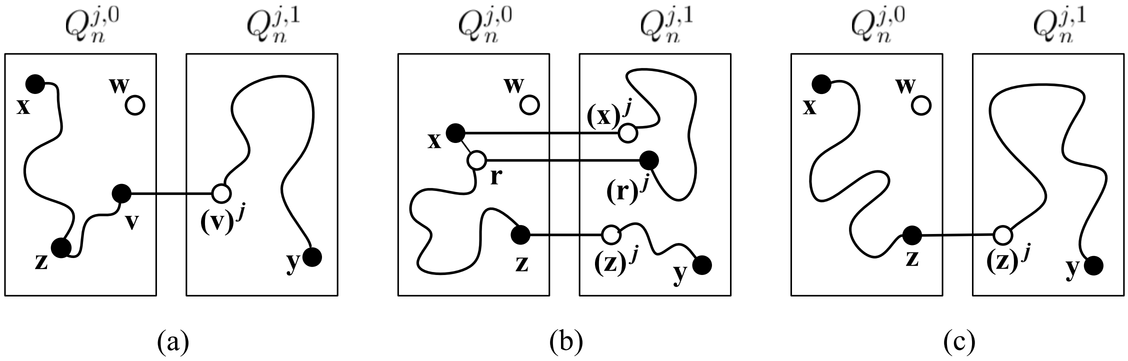

Fig. 3. $\mathbf{z} \in V_1(Q_n^{j,0})$. (a) $l \leq 2^{n-1} - 4$. (b) $l \geq 2^{n-1}$. (c) $l = 2^{n-1} - 2$.

out to be a hamiltonian path of $Q_n - \{\mathbf{w}\}$ joining $\mathbf{x}$ and $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = l$ for any even integer $l \leq 2^{n-1} - 4$. See Figure 3(a) for illustration.

Subcase 2.2: Suppose that $l \geq 2^{n-1}$. Let $\mathbf{r}$ be a vertex of $Q_n^{j,0}$ such that $h(\mathbf{x}, \mathbf{r}) = \mathbf{1}$ and $(\mathbf{r})^j \neq \mathbf{y}$. By inductive hypothesis, $Q_n^{j,0} - \{\mathbf{w}\}$ has a hamiltonian path L between $\mathbf{x}$ and $\mathbf{z}$ such that $d_L(\mathbf{x}, \mathbf{r}) = 1$. For clarity, we write L as $\langle \mathbf{x}, \mathbf{r}, L_1, \mathbf{z} \rangle$. By Lemma 4, there exist two disjoint paths R_1 and R_2 such that R_1 is a path of length $l - 2^{n-1} + 1$ joining $(\mathbf{x})^j$ to $(\mathbf{r})^j$, and R_2 is a path of length $2^n - l - 3$ joining $(\mathbf{z})^j$ to $\mathbf{y}$. Let $H = \langle \mathbf{x}, (\mathbf{x})^j, R_1, (\mathbf{r})^j, \mathbf{r}, L_1, \mathbf{z}, (\mathbf{z})^j, R_2, \mathbf{y} \rangle$. Therefore, H is a hamiltonian path of $Q_n - \{\mathbf{w}\}$ between $\mathbf{x}$ and $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = l$. See Figure 3(b) for illustration.

Subcase 2.3: Suppose that $l = 2^{n-1} - 2$. Since $\mathbf{x}$ and $\mathbf{z}$ are in the same partite set, Lemma 2 ensures that $Q_n^{j,0} - \{\mathbf{w}\}$ has a hamiltonian path L between $\mathbf{x}$ and $\mathbf{z}$. By Lemma 1, $Q_n^{j,1}$ is hamiltonian laceable. Thus $Q_n^{j,1}$ has a hamiltonian path R between $(\mathbf{z})^j$ and $\mathbf{y}$. Let $H = \langle \mathbf{x}, L, \mathbf{z}, (\mathbf{z})^j, R, \mathbf{y} \rangle$. As a result, H is a hamiltonian path of $Q_n - \{\mathbf{w}\}$ between $\mathbf{x}$ and $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = l = 2^{n-1} - 2$. See Figure 3(c) for illustration.

Case 3: Suppose that $\mathbf{z} \in V_0(Q_n^{j,1})$. We consider the following subcases.

Subcase 3.1: Suppose that $l \leq 2^{n-1} - 3$. Let $\mathbf{v}$ be a vertex of $Q_n^{j,0}$ such that $d_{Q_n}(\mathbf{v}, (\mathbf{z})^j) = 2$. By inductive hypothesis, $Q_n^{j,0} - \{\mathbf{w}\}$ has a hamiltonian path L between $\mathbf{x}$ and $\mathbf{v}$ such that $d_L((\mathbf{z})^j, \mathbf{x}) = l - 1$. For clarity, we write L as $\langle \mathbf{x}, L_1, (\mathbf{z})^j, L_2, \mathbf{v} \rangle$. Since $d_{Q_n}(\mathbf{v}, (\mathbf{z})^j) = 2$, path L_2 can be written as $\langle (\mathbf{z})^j, \mathbf{u}, L_3, \mathbf{v} \rangle$, where $\mathbf{u}$ is some vertex of $Q_n^{j,0}$ adjacent to $(\mathbf{z})^j$.

Firstly, we assume that $(\mathbf{u})^j \neq \mathbf{y}$. By Lemma 4, there exist two disjoint paths R_1 and R_2 such that R_1 joins $\mathbf{z}$ to $(\mathbf{u})^j$, R_2 joins $(\mathbf{v})^j$ to $\mathbf{y}$, and $\ell(R_1) + \ell(R_2) = 2^{n-1} - 2$. Let $H = \langle \mathbf{x}, L_1, (\mathbf{z})^j, \mathbf{z}, R_1, (\mathbf{u})^j, \mathbf{u}, L_3, \mathbf{v}, (\mathbf{v})^j, R_2, \mathbf{y} \rangle$. Then H is a hamiltonian path of $Q_n - \{\mathbf{w}\}$ between $\mathbf{x}$ and $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = l$. See Figure 4(a) for illustration.

Secondly, we assume that $(\mathbf{u})^j = \mathbf{y}$. By Lemma 2, $Q_n^{j,1} - \{\mathbf{y}\}$ has a hamiltonian path R between $\mathbf{z}$ and $(\mathbf{v})^j$. Let $H = \langle \mathbf{x}, L_1, (\mathbf{z})^j, \mathbf{z}, R, (\mathbf{v})^j, \mathbf{v}, L_3^{-1}, \mathbf{u}, \mathbf{y} \rangle$. Then H forms a hamiltonian path of $Q_n - \{\mathbf{w}\}$ between $\mathbf{x}$ and $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = l$. See Figure 4(b) for illustration.

Subcase 3.2: Suppose that $l \geq 2^{n-1} + 1$. Let $\mathbf{u} \in V_0(Q_n^{j,0})$ such that $d_{Q_n}(\mathbf{x}, \mathbf{u}) = 1$ and $(\mathbf{u})^j \neq \mathbf{y}$. By inductive hypothesis, $Q_n^{j,0} - \{\mathbf{w}\}$ has a hamiltonian path L between $\mathbf{x}$ and $(\mathbf{z})^j$ such that $d_{Q_n}(\mathbf{x}, \mathbf{u}) = 1$. Thus we can write L as $\langle \mathbf{x}, \mathbf{u}, L_1, (\mathbf{z})^j \rangle$. By Lemma 4, there exist two disjoint paths R_1 and R_2 such that R_1 joins $(\mathbf{x})^j$ to $(\mathbf{u})^j$ with $\ell(R_1) = l - 2^{n-1}$ and R_2 joins $\mathbf{z}$ to $\mathbf{y}$ with $\ell(R_2) = 2^n - l - 2$. Let $H = \langle \mathbf{x}, (\mathbf{x})^j, R_1, (\mathbf{u})^j, \mathbf{u}, L_1, (\mathbf{z})^j, \mathbf{z}, R_2, \mathbf{y} \rangle$. Then H is a hamiltonian path of $Q_n - \{\mathbf{w}\}$ between $\mathbf{x}$ and $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = l$. See Figure 4(c) for illustration.

Subcase 3.3: Suppose that $l = 2^{n-1} - 1$. By Lemma 2, $Q_n^{j,0}$ is hyper-hamiltonian laceable. Since $\mathbf{x}$ and $(\mathbf{z})^j$ belong to the same partite set of Q_n, $Q_n^{j,0} - \{\mathbf{w}\}$ has a hamiltonian path L between $\mathbf{x}$ and $(\mathbf{z})^j$. By Lemma 1, $Q_n^{j,1}$ is hamiltonian

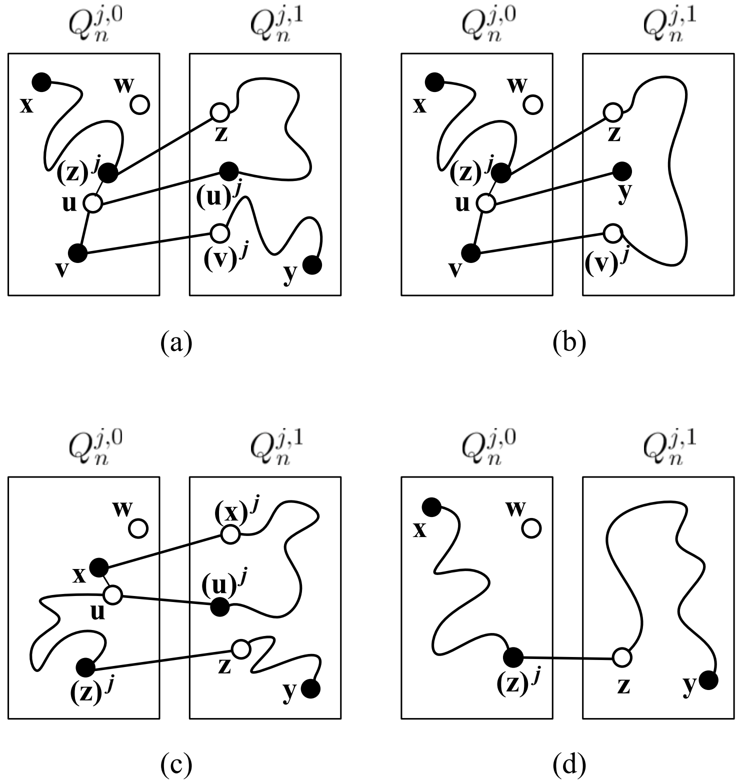

Fig. 4. $\mathbf{z} \in V_0(Q_n^{j,1})$. (a) $l \leq 2^{n-1} - 3$ and $(\mathbf{u})^j \neq \mathbf{y}$. (b) $l \leq 2^{n-1} - 3$ and $(\mathbf{u})^j = \mathbf{y}$. (c) $l \geq 2^{n-1} + 1$. (d) $l = 2^{n-1} - 1$.

laceable. Therefore, $Q_n^{j,1}$ has a hamiltonian path R between $\mathbf{z}$ and $\mathbf{y}$. We set H to be $\langle \mathbf{x}, L, (\mathbf{z})^j, \mathbf{z}, R, \mathbf{y} \rangle$. As a result, H is a hamiltonian path of $Q_n - \{\mathbf{w}\}$ between $\mathbf{x}$ and $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = l = 2^{n-1} - 1$. See Figure 4(d) for illustration.

Case 4: Suppose that $\mathbf{z} \in V_1(Q_n^{j,1})$. Similarly, we consider the following subcases.

Subcase 4.1: Suppose that $l \leq 2^{n-1} - 2$. Let $\mathbf{u} \in V(Q_n^{j,1})$ such that $d_{Q_n}(\mathbf{u}, \mathbf{y}) = 1$ and $(\mathbf{u})^j \neq \mathbf{x}$. By Lemma 3, $Q_n^{j,0} - \{\mathbf{x}, \mathbf{w}\}$ is hamiltonian laceable. Clearly, $(\mathbf{z})^j$ and $(\mathbf{u})^j$ are in different partite sets. Therefore, $Q_n^{j,0} - \{\mathbf{x}, \mathbf{w}\}$ has a hamiltonian path L between $(\mathbf{z})^j$ and $(\mathbf{u})^j$. By Lemma 4, there exist two disjoint paths R_1 and R_2 such that R_1 joins $(\mathbf{x})^j$ to $\mathbf{z}$ with $\ell(R_1) = l - 1$ and R_2 joins $(\mathbf{u})^j$ to $\mathbf{y}$ with $\ell(R_2) = 2^{n-1} - l - 1$. Then we set H to be $\langle \mathbf{x}, (\mathbf{x})^j, R_1, \mathbf{z}, (\mathbf{z})^j, L, (\mathbf{u})^j, \mathbf{u}, R_2, \mathbf{y} \rangle$. As a result, H is a hamiltonian path of $Q_n - \{\mathbf{w}\}$ between $\mathbf{x}$ and $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = l$. See Figure 5(a) for illustration.

Subcase 4.2: Suppose that $l \geq 2^{n-1}$. Let $\mathbf{u} \in V(Q_n^{j,0})$ such that $d_{Q_n}((\mathbf{u})^j, \mathbf{z}) = 1$. By Lemma 2, $Q_n^{j,0} - \{\mathbf{w}\}$ has a hamiltonian path L between $\mathbf{x}$ and $\mathbf{u}$. By Theorem 8, there exists a hamiltonian path R of $Q_n^{j,1}$ joining $(\mathbf{u})^j$ to $\mathbf{y}$ such that $d_R((\mathbf{u})^j, \mathbf{z}) = l - 2^{n-1} + 1$. Then we set H to be $\langle \mathbf{x}, L, \mathbf{u}, (\mathbf{u})^j, \mathbf{z}, R, \mathbf{y} \rangle$. Consequently, H forms a hamiltonian path of $Q_n - \{\mathbf{w}\}$ between $\mathbf{x}$ and $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = l$. See Figure 5(b) for illustration. ■

IV. Conclusion

In this paper, we prove that, for any two vertices $\mathbf{x}$ and $\mathbf{y}$ from one partite set of Q_n ($n \geq 4$) and for any vertex $\mathbf{w}$ from the other partite set, there exists a hamiltonian path H of $Q_n - \{\mathbf{w}\}$ joining $\mathbf{x}$ to $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = l$ for any vertex $\mathbf{z} \in V(Q_n) - \{\mathbf{x}, \mathbf{y}, \mathbf{w}\}$ and for every integer l satisfying both $d_{Q_n}(\mathbf{x}, \mathbf{z}) \leq l \leq 2^n - 2 - d_{Q_n}(\mathbf{z}, \mathbf{y})$ and $2 | (l - d_{Q_n}(\mathbf{x}, \mathbf{z}))$. Specifically, we give the following example to indicate why such a result is not true for Q_3. See Figure 6 for illustration, in which we assume that $\mathbf{x} = 100$, $\mathbf{y} = 010$, $\mathbf{w} = 000$, and $\mathbf{z} = 110$. Clearly, there does not exist any hamiltonian path H in $Q_3 - \{\mathbf{w}\}$ joining $\mathbf{x}$ to $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = 3$.

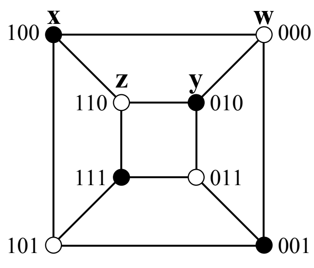

Fig. 6. $Q_3 - \{\mathbf{w}\}$ has no hamiltonian path H between $\mathbf{x}$ and $\mathbf{y}$ such that $d_H(\mathbf{x}, \mathbf{z}) = 3$.

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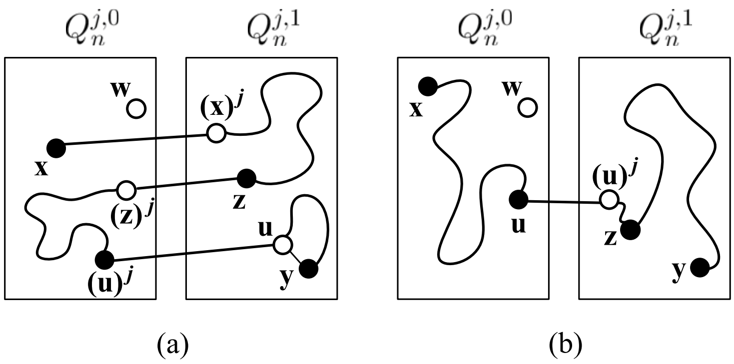

Fig. 5. $\mathbf{z} \in V_1(Q_n^{j,1})$. (a) $l \leq 2^{n-1} - 2$. (b) $l \geq 2^{n-1}$.

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