On the level-crossing probability in a queue with autcorrelated arrivals

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***Abstract*—** In this paper the probability that the queue size is kept below some threshold in time interval of a given length is studied. In particular, a formula for this probability is shown in terms of generating functions. In addition to analytical results, a set of numerical results is presented. These numerical results reveal a surprising phenomenon. Namely, when the arrival process is strongly autocorrelated, the level-crossing probability may depend very little on the system load. As IP traffic is often strongly autocorrelated, it is likely that this phenomenon may be observed in queues of packets in Internet routers.

***Keywords*—** level-crossing probability, single-server queue

I. Introduction

In this paper we deal with the single-server queue with infinite waiting room [9]. Let us suppose that it is important that the queue length is kept below some level L as long as possible and that queues longer than L, although allowed, are very undesirable.

The purpose of this paper is to calculate the probability that in the time interval of length t the queue size does not exceed the level L.

This study can be motivated by the observation that in most applications of queueing systems the queue size is desired to be small. This is especially true when considering traffic buffering in packet-oriented telecommunication networks (e.g. Internet). It is well known that in this (dominant nowadays) networking technology the packets are queued (buffered) in network nodes. Keeping short queues in network nodes is crucial for maintaining low end-to-end delays and, therefore, for providing satisfactory level of network services.

To make the analytical results widely applicable, a general queueing model is assumed. In particular, it includes general type of the service time distribution and autocorrelated arrival process.

The autocorrelated structure of the arrival process is a must when modeling telecommunication traffic, as it often shows strongly correlated and long-range dependent behaviour [14], [10], [18]. On the other hand, it is well known that passing over the autocorrelation in the arrival stream leads to serious and optimistic overestimation of the queueing characteristics [11].

Therefore, to enable modeling of the interarrival times autocorrelation, the Markovian arrival process (MAP, [16], [4]) is used herein. It is quite flexible (yet analytically tractable) traffic model that allows for not only precise fitting of the autocorrelation function, but also for fitting of the mean traffic rate, higher moments and the marginal distribution at the same time [19], [13], [2], [3]. The MAP is also widely used in telecommunication traffic modeling [17], [12], [20], [21]).

This is extended version of the paper [1] presented during MATH'08 conference, Puerto De La Cruz, December 15-17, 2008.

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II. Model description

Let us denote by $N(t)$ the total number of arrivals in $(0, t]$ and let $J(t)$ be the state of a continuous-time Markov chain. The Markovian arrival process (MAP) is a 2-dimensional Markov process $(N(t), J(t))$ on the state space $\{(i, j) : i \geq 0, 1 \leq j \leq m\}$ with a generator matrix Q in the following form:

$$Q = \begin{bmatrix} D_0 & D_1 & 0 & 0 & 0 & \cdot\cdot \\ 0 & D_0 & D_1 & 0 & 0 & \cdot\cdot \\ 0 & 0 & D_0 & D_1 & 0 & \cdot\cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot \end{bmatrix}.$$

Here D_0 and D_1 are matrices of size $m \times m$, D_1 is nonnegative, D_0 has negative diagonal elements and nonnegative off-diagonal elements and the matrix

$$D = D_0 + D_1$$

is an irreducible infinitesimal generator. We assume also that $D \neq D_0$.

As for the model of the queueing system, we consider herein a single-server queue [9] whose arrival process is given by a MAP. The service time is distributed according to a distribution function $F(\cdot)$, which is not further specified, and the standard independence assumptions are made. The capacity of the system (waiting room) is infinite. The queueing discipline is irrelevant in this study, therefore FCFS (First Come First Served) or FCLS (First Come Last Served) can be assumed.

Here and subsequently $\mathbf{P}(\cdot)$ stands for the probability, $\mathbf{0}$ stands for the $m \times m$ matrix of zeroes, I stands for the $m \times m$ identity matrix, while $\mathbf{1}$ denotes the column vector of 1's.

Moreover, we will use the following notation:

$$\begin{aligned}
\mu_i &= -(D_0)_{ii}, \\
p_{ij} &= \frac{1}{\mu_i}(D_0)_{ij} \quad \text{for} \quad i \neq j, \quad p_{ii} = 0, \\
q_{ij} &= \frac{1}{\mu_i}(D_1)_{ij}, \\
y(s) &= ((s+\mu_1)^{-1}, \dots, (s+\mu_m)^{-1})^T, \\
U(s) &= \left[\frac{\mu_i p_{ij}}{s+\mu_i}\right]_{i,j}, \\
V(s) &= \left[\frac{\mu_i q_{ij}}{s+\mu_i}\right]_{i,j}.
\end{aligned}$$

III. Level-crossing probability

Let $X(t)$ stands for the queue size at the moment t (including service position). Let the initial queue size be n and the initial state of the Markov chain J be i. Let $B(t) = B(L,t)$, $L > n$, denotes event that in time interval $(0,t]$ the queue length is smaller than L, i.e. for every $t_0 \in (0,t]$ is $X(t_0) < L$.

The main result of this article is presented by means of the following transform:

$$\begin{aligned}
\alpha_{n,i}(s) &= \int_0^\infty e^{-st}\mathbf{P}(B(t)|X(0)=n, J(0)=i)dt, \\
\alpha_n(s) &= (\alpha_{n,1}(s), \alpha_{n,2}(s), \dots, \alpha_{n,m}(s))^T.
\end{aligned}$$

Theorem 1. *For $n < L$ it holds true that:*

$$\alpha_n(s) = G_{L-n}(s)T_L^{-1}(s)c_L(s) - h_{L-n}(s), \tag{1}$$

where the sequences $G_k(s)$, $T_k(s)$, $c_k(s)$, $h_k(s)$ have the following generating functions:

$$\begin{aligned}
G(z,s) &= \sum_{k=0}^{\infty} z^k G_k(s) \\
&= z(A(z,s) - Iz)^{-1}A(z,s), \qquad (2) \\
A(z,s) &= \int_0^\infty e^{-st}e^{D(z)t}dF(t), \\
D(z) &= D_0 + zD_1,
\end{aligned}$$

$$\begin{aligned}
h(z,s) &= \sum_{k=1}^{\infty} z^k h_k(s) \\
&= z(A(z,s) - Iz)^{-1}d(z,s), \qquad (3) \\
d(z,s) &= \frac{z}{1-z}\int_0^\infty e^{-st}e^{D(z)t}(1-F(t))dt \cdot \mathbf{1}, \\
c(z,s) &= \sum_{k=1}^{\infty} z^k c_k(s) \\
&= (U(s) + zV(s) - I)h(z,s) - \frac{z}{1-z}y(s), \qquad (4) \\
T(z,s) &= \sum_{k=1}^{\infty} z^k T_k(s) \\
&= (U(s) + zV(s) - I)G(z,s). \qquad (5)
\end{aligned}$$

Proof. Firstly, according to the total probability formula we have for $1 \le i \le m$, $0 < n < L$:

$$\begin{aligned}
&\mathbf{P}(B(t)|X(0)=n, J(0)=i) \\
&= \sum_{j=1}^{m}\sum_{k=0}^{L-n-1}\int_0^t \mathbf{P}(B(t-u)|X(0)=n+k-1, \\
&\qquad J(0)=j)P_{i,j}(k,u)dF(u) \\
&+ (1-F(t))\sum_{j=1}^{m}\sum_{k=0}^{L-n-1}P_{i,j}(k,t). \qquad (6)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{P}(B(t)|X(0)=0, J(0)=i) \\
&= \sum_{j=1}^{m}\int_0^t \mathbf{P}(B(t-u)|X(0)=0, J(0)=i) \\
&\qquad \cdot p_{ij}\mu_i e^{-\mu_i u}du \\
&+ \sum_{j=1}^{m}\int_0^t \mathbf{P}(B(t-u)|X(0)=1, J(0)=i) \\
&\qquad \cdot q_{ij}\mu_i e^{-\mu_i u}du + e^{-\mu_i t}, \qquad (7)
\end{aligned}$$

where $P_{i,j}(n,t)$ denotes the counting function for the MAP:

$$\begin{aligned}
&P_{i,j}(n,t) \\
&= \mathbf{P}(N(t)=n, J(t)=j|N(0)=0, J(0)=i).
\end{aligned}$$

We proceed then by applying transforms to (6), (7) and writing results in matrix form:

$$\begin{aligned}\alpha_n(s) &= \sum_{k=0}^{L-n-1} A_k(s)\alpha_{n+k-1}(s) \\ &\quad +d_{L-n}(s), \qquad 0 < n < L, \end{aligned} \tag{8}$$

$$\alpha_0(s) = U(s)\alpha_0(s) + V(s)\alpha_1(s) + y(s), \tag{9}$$

where

$$A_k(s) = [a_{k,i,j}(s)]_{i,j} \,,$$

$$a_{k,i,j}(s) = \int_0^\infty e^{-st} P_{i,j}(k,t) dF(t),$$

$$d_n(s) = (d_{n,1}(s), \ldots, d_{n,m}(s))^T,$$

$$d_{n,i}(s) = \sum_{j=1}^{m}\sum_{k=0}^{n-1}\int_0^\infty e^{-st} P_{i,j}(k,t)(1-F(t))dt.$$

The system (8) i (9) can be solved in the same way as shown in [5]. We obtain:

$$\begin{aligned}\alpha_n(s) = &\sum_{k=0}^{L-n} R_{L-n-k}(s)A_k(s)T_L^{-1}(s)c_L(s) \\ &- \sum_{k=1}^{L-n} R_{L-n-k}(s)d_k(s), \quad 0 \le n \le L-1, \end{aligned} \tag{10}$$

with

$$R_0(s) = \mathbf{0},$$

$$R_1(s) = A_0^{-1}(s),$$

$$R_{k+1}(s) = R_1(s)\left(R_k(s) - \sum_{i=0}^{k} A_{i+1}(s)R_{k-i}(s)\right),$$

$$k \ge 1, \tag{11}$$

$$\begin{aligned}T_L(s) &= U(s)\sum_{i=0}^{L} R_{L-i}(s)A_i(s) \\ &\quad +V(s)\sum_{i=0}^{L-1} R_{L-1-i}(s)A_i(s) \\ &\quad -\sum_{k=0}^{L} R_{L-k}(s)A_k(s), \end{aligned} \tag{12}$$

$$\begin{aligned}c_L(s) &= U(s)\sum_{i=1}^{L-1} R_{L-i}(s)d_i(s) \\ &\quad +V(s)\sum_{i=1}^{L-2} R_{L-1-i}(s)d_i(s) \\ &\quad -\sum_{k=1}^{L} R_{L-k}(s)d_k(s) - y(s). \end{aligned} \tag{13}$$

Note that the sequence $R_k(s)$ is called a potential for the sequence $A_k(s)$ and is also useful in finding other characteristics of queueing systems (see, for instance, [6], [7], [8]).

Defining

$$G_n(s) = \sum_{k=0}^{n} R_{n-k}(s)A_k(s), \tag{14}$$

$$h_n(s) = \sum_{k=1}^{n} R_{n-k}(s)d_k(s), \tag{15}$$

and using (10) we obtain (1).

Now, following [15], the generating function for the sequence $A_k(s)$ has the form:

$$\begin{aligned}A(z,s) &= \sum_{k=0}^{\infty} A_k(s)z^k \\ &= \int_0^\infty e^{-st}e^{D(z)t}dF(t),\end{aligned}$$

Similarly, it is easy to obtain the generating function for $d_n(s)$:

$$\begin{aligned}d(z,s) &= \sum_{k=1}^{\infty} d_k(s)z^k \\ &= \frac{z}{1-z}\int_0^\infty e^{-st}e^{D(z)t}(1-F(t))dt \cdot \mathbf{1}.\end{aligned}$$

From (11) it follows that

$$\begin{aligned}R(z,s) &= \sum_{k=0}^{\infty} R_k(s)z^k \\ &= z(A(z,s) - Iz)^{-1}.\end{aligned}$$

Observing that in (14) and (15) we have convolutions of sequences, we obtain

$$G(z,s) = R(z,s)A(z,s),$$

$$h(z,s) = R(z,s)d(z,s),$$

and, as a consequence, formulas (2), (3).

From (12) and (14) it follows that

$$\begin{aligned}&T_L(s) \\ &\quad = U(s)G_L(s) + V(s)G_{L-1}(s) - G_L(s),\end{aligned} \tag{16}$$

which proves (5), while from (13) and (15) it follows that

$$\begin{aligned}&c_L(s) \\ &\quad = U(s)h_L(s) + V(s)h_{L-1}(s) - h_L(s) - y(s),\end{aligned} \tag{17}$$

which yields (4).

This finishes the proof of Theorem 1.

□

IV. NUMERICAL EXAMPLES

A. Example 1

We start numerical experiments with the following MAP parameterization:

$$D_0 = \begin{bmatrix} -1000 & 10 \\ 1 & -10 \end{bmatrix}, \tag{18}$$

$$D_1 = \begin{bmatrix} 986 & 4 \\ 1 & 8 \end{bmatrix}. \tag{19}$$

This parameterization was chosen so that the resulting arrival process is strongly autocorrelated – its autocorrelation function is presented in Fig. 1.

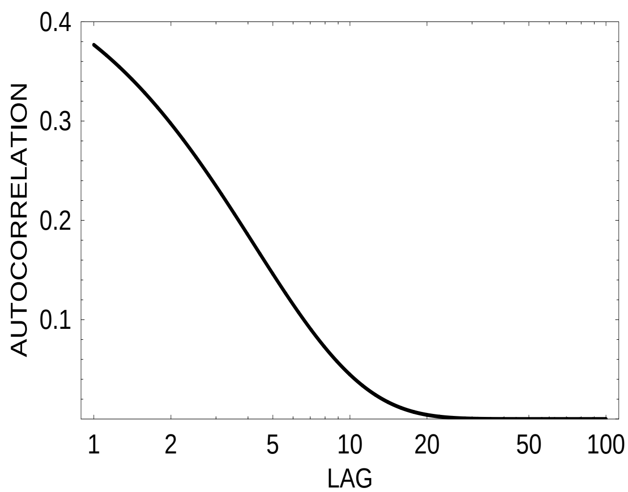

Fig. 1. Autocorrelation of the MAP defined in (18) and (19).

The average arrival rate of this MAP is

$$\lambda = 131.62,$$

while the stationary distribution of the underlying Markov chain:

$$\pi = (0.125, 0.875).$$

The level that we do not want to be crossed is set to

$$L = 10.$$

It is assumed that the service time is constant and equal to d. Therefore, manipulating d we can easily manipulate the system load

$$\rho = \lambda d.$$

In the reminder of this section the behaviour of the function $r_{n,i}(t)$ defined as

$$r_{n,i}(t) = \mathbf{P}(B(t)|X(0) = n, J(0) = i)$$

will be studied for different values of the initial queue size, n, the initial modulating state, i, and the system load, ρ. We will use also the symbol $r_n^*(t)$ in the case when the initial modulating state is distributed according to π, namely:

$$r_n^*(t) = \pi \cdot (r_{n,1}(t), r_{n,2}(t))^T.$$

In Figs. 2, 3 and 4 the dependence of $r_{n,i}(t)$ on the initial queue size is depicted. Namely, in Fig. 2 the results for $\rho = 0.75$ are shown using linear plot while in Fig. 3 – using a logarithmic plot. In Fig. 4 the results for $\rho = 0.99$ are presented. We see that the values of $r_{n,i}(t)$ depend strongly on the initial queue size and that the level-crossing probability decreases exponentially with t.

In Fig. 5 the dependence of $r_{n,i}(t)$ on the initial modulating state is shown. We see that for the MAP parameterization (18) and (19) the level crossing probability depends strongly on $J(0)$.

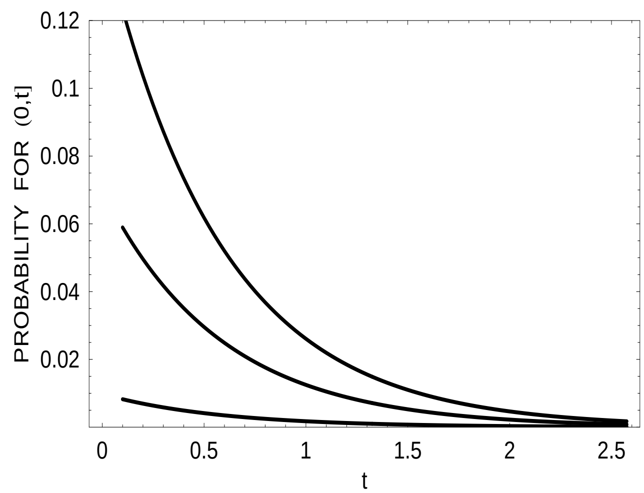

Fig. 2. Probability that in $(0, t]$ the queue size is below 10 for initial queue sizes: 9, 5 and 0, counting from the bottom. The MAP defined in (18) and (19), $\rho = 0.75$, $i = 1$.

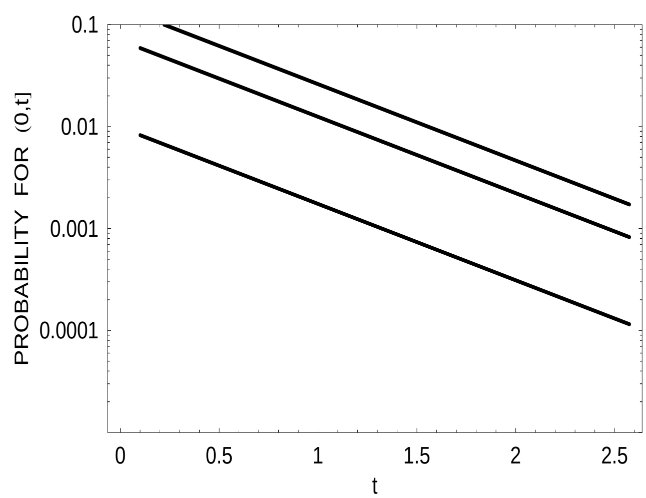

Fig. 3. Probability that in $(0, t]$ the queue size is below 10 for initial queue sizes: 9, 5 and 0, counting from the bottom. The MAP defined in (18) and (19), $\rho = 0.75$, $i = 1$.

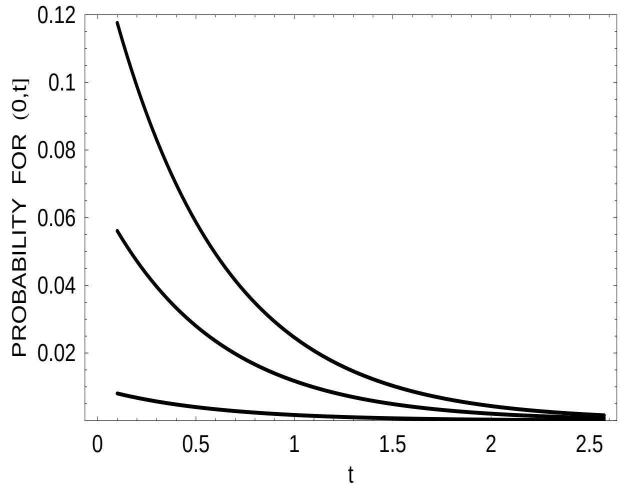

Fig. 4. Probability that in $(0,t]$ the queue size is below 10 for initial queue sizes: 9, 5 and 0, counting from the bottom. The MAP defined in 18) and (19), $\rho = 0.99$, $i = 1$.

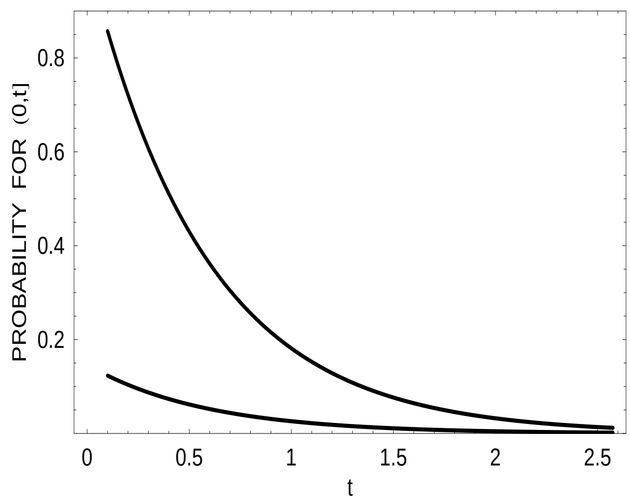

Fig. 5. Probability that in $(0,t]$ the queue size is below 10 for $i = 1$ (lower curve) and $i = 2$ (upper curve). The MAP defined in (18) and (19), $\rho = 0.75$, $n = 0$.

In Fig. 6 the dependence of $r_{n,i}(t)$ on the system load is presented. These results are quite surprising – all the curves are almost identical, even for such a different loads as 0.75 and 0.99. For instance, for $\rho = 0.75$ we have:

$$r_{0,1}(1) = 0.0261, \tag{20}$$

while for $\rho = 0.99$:

$$r_{0,1}(1) = 0.0246. \tag{21}$$

It is important to check that the phenomenon presented in Fig. 6 has nothing to do with the choice of the initial modulating state. Indeed, for the second modulating state the curves are also almost identical (see Fig. 7). Again, for $\rho = 0.75$ we have

$$r_{0,2}(1) = 0.1814, \tag{22}$$

while for $\rho = 0.99$

$$r_{0,2}(1) = 0.1789. \tag{23}$$

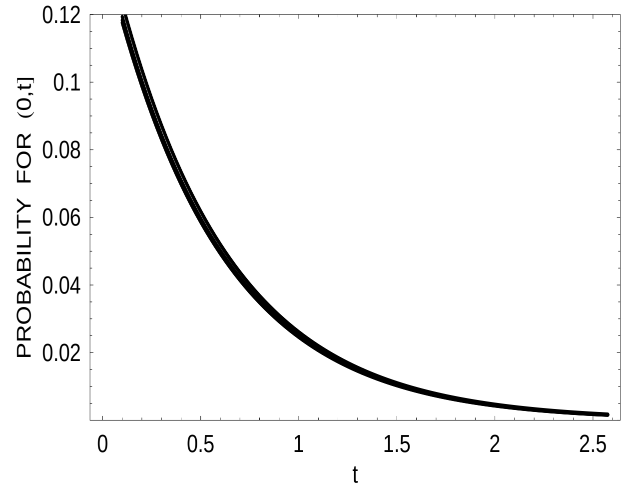

Fig. 6. Probability that in $(0,t]$ the queue size is below 10 for $\rho = 0.99$, 0.95, 0.95 and 0.75. The MAP defined in (18) and (19), $n = 0$, $i = 1$. All the curves are almost identical.

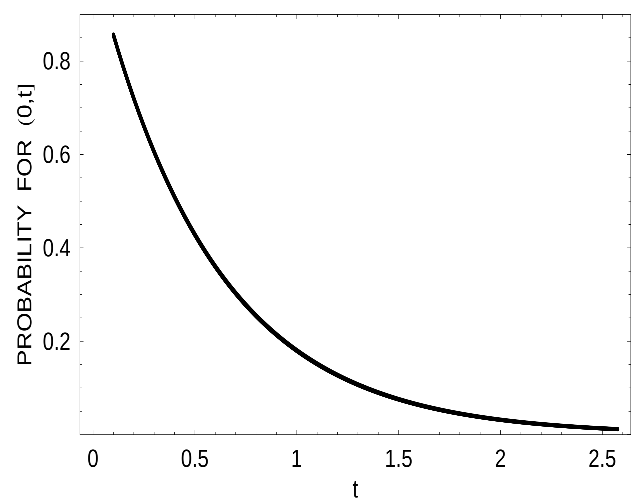

Fig. 7. Probability that in $(0,t]$ the queue size is below 10 for $\rho = 0.99$, 0.95, 0.95 and 0.75. The MAP defined in (18) and (19), $n = 0$, $i = 2$. All the curves are almost identical.

A detailed comparison of the level-crossing probabilities for $\rho = 0.75$ and 0.99 and the stationary distribution of $J(0)$ is presented in Table I. In Tab. II the dependence of $r_0^*(1)$ on the system load is shown.

The phenomenon shown in Figs. 6, 7 and Tabs. I, II is likely to be connected with the strong autocorrelation of the arrival process. In the next example we will check the behaviour of $r_{n,i}(t)$ for a very weakly correlated MAP.

t	$r_0^*(t)$ for $\rho = 0.75$	$r_0^*(t)$ for $\rho = 0.99$
0.1	0.7653881	0.7631767
0.2	0.6440878	0.6414009
0.3	0.5420114	0.5390561
0.4	0.4561123	0.4530419
0.5	0.3838267	0.3807525
0.6	0.3229970	0.3199979
0.7	0.2718078	0.2689376
0.8	0.2287311	0.2260247
0.9	0.1924813	0.1899592
1.0	0.1619765	0.1596484
1.1	0.1363061	0.1341742
1.2	0.1147040	0.1127648
1.3	0.0965255	0.0947715
1.4	0.0812279	0.0796493
1.5	0.0683547	0.0669401
1.6	0.0575217	0.0562589
1.7	0.0484056	0.0472819
1.8	0.0407341	0.0397374
1.9	0.0342785	0.0333967
2.0	0.0288460	0.0280678
2.1	0.0242744	0.0235892
2.2	0.0204273	0.0198252
2.3	0.0171899	0.0166618
2.4	0.0144656	0.0140031
2.5	0.0121731	0.0117687
2.6	0.0102439	0.0098908
2.7	0.0086204	0.0083126
2.8	0.0072542	0.0069862
2.9	0.0061045	0.0058714
3.0	0.0051371	0.0049346
3.1	0.0043229	0.0041472
3.2	0.0036378	0.0034854
3.3	0.0030613	0.0029293
3.4	0.0025761	0.0024619
3.5	0.0021678	0.0020690
3.6	0.0018243	0.0017389
3.7	0.0015351	0.0014614
3.8	0.0012918	0.0012282
3.9	0.0010871	0.0010322
4.0	0.0009148	0.0008675
4.1	0.0007698	0.0007291
4.2	0.0006478	0.0006127
4.3	0.0005451	0.0005150
4.4	0.0004587	0.0004328
4.5	0.0003860	0.0003637
4.6	0.0003248	0.0003057
4.7	0.0002733	0.0002569
4.8	0.0002300	0.0002159
4.9	0.0001936	0.0001814
5.0	0.0001629	0.0001525

TABLE I

PROBABILITIES $r_0^*(t)$ FOR $\rho = 0.75$ AND 0.99 IN EXAMPLE 1.

ρ	$r_0^*(1)$
0.05	0.9999636
0.10	0.8926249
0.15	0.3337932
0.20	0.2232732
0.25	0.1953409
0.30	0.1832879
0.35	0.1766358
0.40	0.1724321
0.45	0.1695390
0.50	0.1674275
0.55	0.1658192
0.60	0.1645534
0.65	0.1635308
0.70	0.1626867
0.75	0.1619765
0.80	0.1613685
0.85	0.1608396
0.90	0.1603724
0.95	0.1599542
1.00	0.1595755

TABLE II

PROBABILITIES $r_0^*(1)$ FOR DIFFERENT VALUES OF ρ IN EXAMPLE 1.

B. Example 2

In the second example we use the following MAP parameterization:

$$D_0 = \begin{bmatrix} -261.540 & 153.847 \\ 46.154 & -184.616 \end{bmatrix}, \tag{24}$$

$$D_1 = \begin{bmatrix} 46.154 & 61.539 \\ 15.385 & 123.077 \end{bmatrix}. \tag{25}$$

The average arrival rate of this MAP is the same as in Example 1, namely

$$\lambda = 131.62,$$

while the stationary distribution of the underlying Markov chain is now:

$$\pi = (0.222222, 0.777777).$$

The level that we do not want to be crossed is again set to

$$L = 10.$$

The resulting MAP is now weakly correlated – its autocorrelation function is presented in Fig. 8.

In Figs. 9, 10 and 11 the dependence of $r_{n,i}(t)$ on the initial queue size is shown. What was to be expected, the values of $r_{n,i}(t)$ depend strongly on the initial queue size.

In Fig. 12 the dependence of $r_{n,i}(t)$ on the system load is presented. These results are quite different compared to those shown in Fig. 6. Now all the curves have different shapes and this behaviour is typical for queueing systems with such different loads. For $\rho = 0.75$ we have now:

$$r_{0,1}(1) = 0.8661,$$

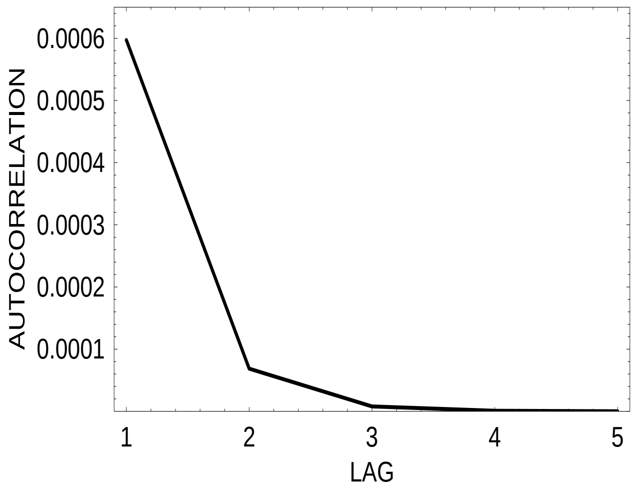

Fig. 8. Autocorrelation of the MAP defined in (24) and (25).

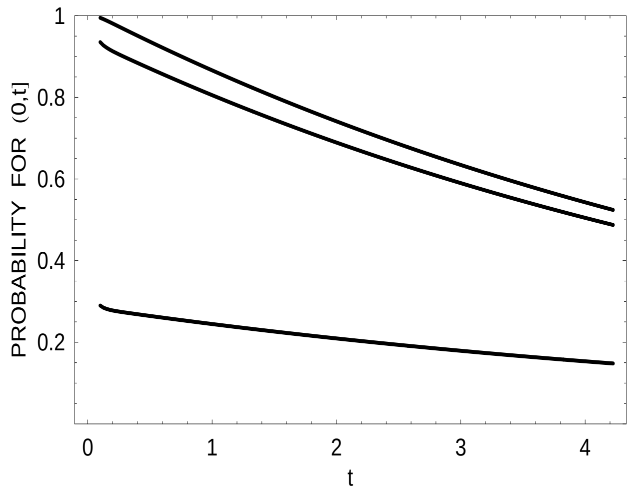

Fig. 9. Probability that in $(0, t]$ the queue size is below 10 for initial queue sizes: 9, 5 and 0, counting from the bottom. The MAP defined in (24) and (25), $\rho = 0.75$, $i = 1$.

while for $\rho = 0.99$:

$$r_{0,1}(1) = 0.2203.$$

The difference between these two numbers is to be compared to the difference between (20) and (21).

A similar behaviour we can observe for $i = 2$. Namely, for $\rho = 0.75$ we have:

$$r_{0,2}(1) = 0.8660,$$

while for $\rho = 0.99$:

$$r_{0,1}(1) = 0.2200.$$

A detailed comparison of the level-crossing probabilities for $\rho = 0.75$ and 0.99 and the stationary distribution of $J(0)$ is given in Table III, while in Tab. IV the dependence of $r_0^*(1)$ on the system load is presented.

t	$r_0^*(t)$ for $\rho = 0.75$	$r_0^*(t)$ for $\rho = 0.99$
0.1	0.99477500	0.9656587
0.2	0.98078737	0.8475935
0.3	0.96576422	0.7214380
0.4	0.95085467	0.6098449
0.5	0.93616447	0.5147387
0.6	0.92170020	0.4343094
0.7	0.90745933	0.3664187
0.8	0.89343847	0.3091352
0.9	0.87963424	0.2608060
1.0	0.86604330	0.2200322
1.1	0.85266235	0.1856328
1.2	0.83948814	0.1566114
1.3	0.82651748	0.1321271
1.4	0.81374723	0.1114706
1.5	0.80117429	0.0940435
1.6	0.78879561	0.0793410
1.7	0.77660818	0.0669370
1.8	0.76460906	0.0564722
1.9	0.75279534	0.0476434
2.0	0.74116414	0.0401950
2.1	0.72971266	0.0339110
2.2	0.71843811	0.0286094
2.3	0.70733776	0.0241366
2.4	0.69640891	0.0203632
2.5	0.68564892	0.0171796
2.6	0.67505519	0.0144938
2.7	0.66462513	0.0122279
2.8	0.65435622	0.0103162
2.9	0.64424598	0.0087034
3.0	0.63429194	0.0073427
3.1	0.62449171	0.0061947
3.2	0.61484289	0.0052263
3.3	0.60534315	0.0044092
3.4	0.59599019	0.0037199
3.5	0.58678174	0.0031383
3.6	0.57771557	0.0026477
3.7	0.56878948	0.0022337
3.8	0.56000130	0.0018845
3.9	0.55134890	0.0015899
4.0	0.54283019	0.0013413
4.1	0.53444309	0.0011316
4.2	0.52618559	0.0009547
4.3	0.51805567	0.0008054
4.4	0.51005136	0.0006795
4.5	0.50217072	0.0005733
4.6	0.49441184	0.0004836
4.7	0.48677285	0.0004080
4.8	0.47925188	0.0003442
4.9	0.47184712	0.0002904
5.0	0.46455676	0.0002450

TABLE III
PROBABILITIES $r_0^*(t)$ FOR $\rho = 0.75$ AND 0.99 IN EXAMPLE 2.

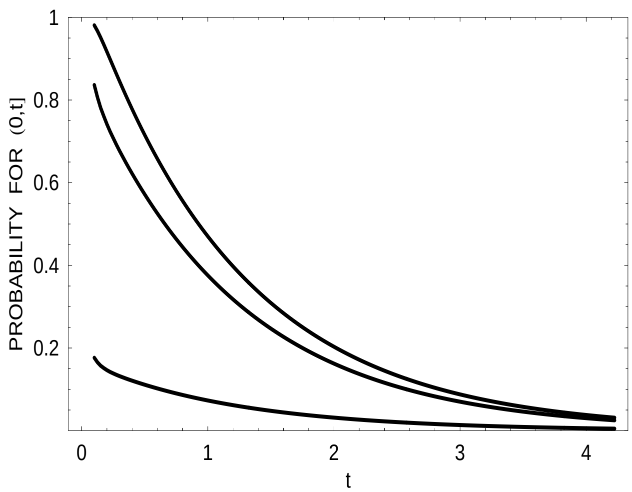

Fig. 10. Probability that in $(0,t]$ the queue size is below 10 for initial queue sizes: 9, 5 and 0, counting from the bottom. The MAP defined in (24) and (25), $\rho = 0.90$, $i = 1$.

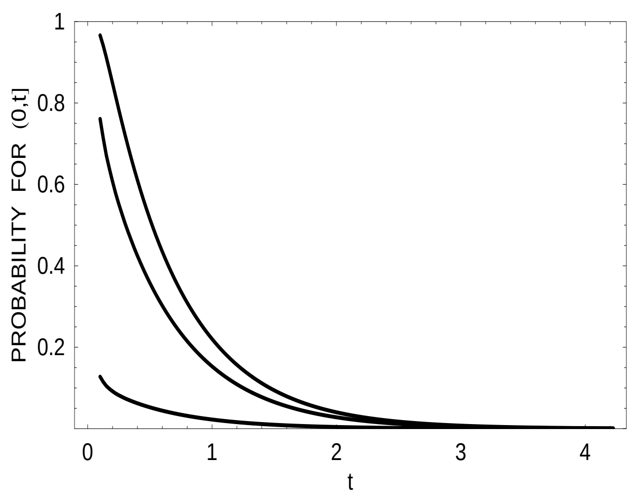

Fig. 11. Probability that in $(0,t]$ the queue size is below 10 for initial queue sizes: 9, 5 and 0, counting from the bottom. The MAP defined in (24) and (25), $\rho = 0.99$, $i = 1$.

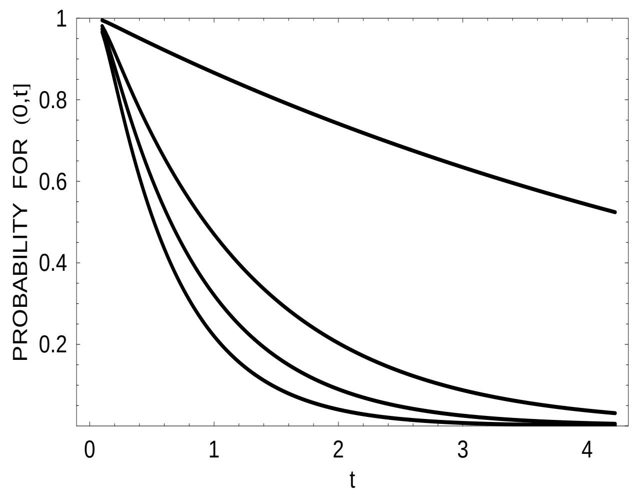

Fig. 12. Probability that in $(0,t]$ the queue size is below 10 for $\rho = 0.99$, 0.95, 0.95 and 0.75, counting form the bottom. The MAP defined in (24) and (25), $n = 0$, $i = 1$.

ρ	$r_0^*(1)$
0.30	0.9999968
0.35	0.9999802
0.40	0.9999022
0.45	0.9995964
0.50	0.9985695
0.55	0.9955519
0.60	0.9877046
0.65	0.9695794
0.70	0.9325183
0.75	0.8660433
0.80	0.7627381
0.85	0.6252880
0.90	0.4698837
0.95	0.3206925
1.00	0.1983160

TABLE IV

PROBABILITIES $r_0^*(1)$ FOR DIFFERENT VALUES OF ρ IN EXAMPLE 2.

V. CONCLUSIONS

In this paper a study on the level-crossing probability in a queue with a complex arrival process is presented. It was motivated by packet buffering processes in Internet routers, but the general service time distribution and flexible arrival process make the results applicable in other areas as well.

The main result is a formula for the level-crossing probability (Theorem 1) presented by means of the Laplace transform and generating functions.

In addition to analytical results, two numerical examples based on two different MAP parameterizations are presented. These examples reveal a surprising behaviour of the level-crossing probability. Namely, it may sometimes depend very little on the system load. This effect is supposed to be connected with the autocorrelated structure of the arrival process and may have some important practical consequences. For instance, it is well known that IP traffic is often strongly autocorrelated. Therefore, the level-crossing probability in an Internet router may depend very little on the router's load, or, equivalently, on the packet transmission time (link speed).

ACKNOWLEDGEMENTS

This work was supported by MNiSW under grant N N516 381134.

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