On Performance Deviation of Binary Search Tree Searches from the Optimal Search Tree Search Structures

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***Abstract*—** Binary Search Trees are a frequently used data structure for rapid access to the stored data. Data structures like arrays, vectors and linked lists are limited by the trade-off between the ability to perform a fast search and resize easily. They are an alternative that is both dynamic in size and easily searchable. Due to efficiency reason, complete and nearly complete binary search trees are of particular significance. This paper addresses the performance analysis and measurement, collectively known as the Performance in binary search tree search applications. Performance measurement is equally significant asides from the performance analysis to learn more about the deviation from optimality. To estimate this deviation, new performance criteria for the binary search trees are presented. A multi-key search algorithm is proposed and the related analysis followed. The algorithm is capable of searching for multiple key elements in the same execution, sacrificing some optimality in the timing consideration. This helps in pruning a subtree structure out of a given binary search tree for further processing.

***Keywords*—** Complete Binary Search Tree, Nearly Complete Binary Search Tree, Performance Criteria, Sparsity Factor, Density Factor, Multi-Key Search, Search-tree Pruning.

I. Introduction

Efficient access to the stored data is a mainstream reason for the choice of a good data structure (DS). To provide efficient access, the DS may need to store additional information known as the overhead. Therefore, a major objective of a DS is to keep the overhead minimum while allowing maximum access to the stored data. This paper is concerned with the analysis of binary search trees (BSTs) as data structures of choice with several performance criteria. The deviation from the optimality for using the BSTs are demonstrated using performance measurement results.

BSTs and the related applications are studied extensively in the literature. Among the most notable contributions, [1] has studied the height, size performance of a class of BSTs in dictionary application. In [2], an application of the BSTs in Neural Networks is presented. This research paper deals with the general performance in BST search applications. A new algorithm in searching for multiple number of nodes in the same execution is also proposed. The multiple key BST search helps prune a subtree structure from an existing BST for further processing. Performance measurement of the proposed multi-key BST search algorithm is also presented.

The results in this paper are both theoretical and applied in nature. The performance graphs are obtained using the common high level language compilers running on multiple platforms. A number of performance criteria are addressed. Finally, future research directions are outlined.

The remainder of this paper is structured as follows. In Section II, terms and notations used in this paper are introduced. Some new concepts are also defined. Section III considers performance of the BSTs using the criteria introduced in section II. Section IV introduces the Multiple Key BST Search algorithm. This section also incorporates the related analysis. Section V is based on the search-based performance of the BSTs. Section VI addresses the practical performance issues. Section VII outlines future research avenues.

II. Terminology and Notations

Following notations are used all throughout this paper.
n: Total number of nodes.
T_r: A binary search tree, which is abbreviated as, BST.
l: Number of leaves.
n_i: Internal (interior) node count.
n_e: Number of external nodes.
h: Height of the BST.
C_{n_i}: Cost for a successful search in a BST.
C_{n_e}: Cost for an unsuccessful search.
I: Internal path length.
E: External path length.
sf: Sparsity factor.
df: Density Factor.
L: Loss in capacity factor.
Special terms and concepts are presented by combining meaningful indices with the corresponding notation. Some useful definitions are presented next.
Deviation in Height, h_{dev}: The deviation in height, h_{dev} is the deviation of the actual height, h from the optimal possible height, h_o. This is expressed in % as follows:
$h_{dev} = \frac{h-h_o}{h_o} \times 100\%$.
Sparsity Factor: Justifies the relative sparsity of an actual BST in comparison to a full, and complete BST with the same height, h. Mathematically, Sparsity Factor, $sf = \frac{n_{max}-n}{n_max} \times 100\%$. Here, n_{max} = maximum possible number of records that may be accommodated in a complete BST with the actual height, $h = (2^{(h+1)} - 1)$, and n = the actual number of records currently present.
Density Factor: This determines the relative density of an actual BST in comparison to a linear slim BST having the same height, h. This is defined mathematically as, $df = \frac{n-n_{min}}{n_{min}} \times 100\%$. Here, n_{min} = the minimum number of records in a slim BST with the actual height, $h = (h + 1)$.

III. Performance with the Sparsity and the Density Factors

It is always desired that a constructed BST be as dense as possible approaching the complete or the nearly complete

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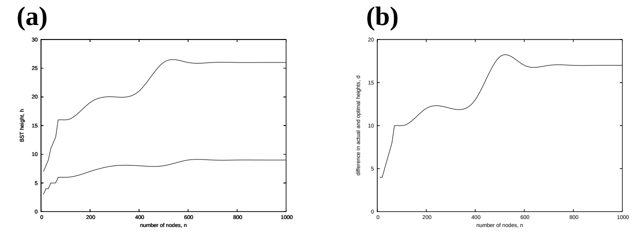
BST structure, and as less sparse as is feasible. Following result holds true in this context.

Theorem 1: The maximum height deviation from a linear sparse BST to a nearly complete bushy BST having n nodes is: $h_{diff_{max}}$ = $(n - log_2(n+1))$, and the corresponding minimum possible height deviation is: $h_{diff_{min}}$ = $(n-1) - log_2 n$, and the difference between these two extreme deviations is: $log_2(2 - \frac{2}{(n+1)})$.

Proof: For a linear skinny tree, there will be exactly 1 node at each level. Since the node counting starts at the root record with level 0, therefore, $(h_s + 1) = n$. This provides, $h_s = (n-1)$. Suppose that there are k records at the last level h. In that event, $(2^0 + 2^1 + \ldots + 2^{h-1}) + k = n$, this means, $\frac{2^h - 1}{(2-1)} + k = n$. Therefore, $2^h - 1 = (n - k)$, this provides, 2^h = $(n + 1 - k)$. Hence, $h = log_2(n+1-k)$. For the minimum deviation in height, there is only 1 record at level h. Therefore, $k = 1$, and $h_{b_{max}}$ = $log_2 n$. Hence, $h_{diff_{min}}$ = $(h_s - h_{b_{max}})$ = $(n-1) - log_2 n$. For the maximum height deviation, there are 2^h records at level h. Therefore, $k = 2^h$, and 2^h = $(n + 1 - 2^h)$, which provides, 2^{h+1} = $(n+1)$. This yields, $(h+1)$ = $log_2(n+1)$, or h = $log_2(n+1) - 1$. Therefore, $h_{diff_{max}}$ = $(h_s - h_{b_{min}})$ = $(n-1) - (log_2(n+1) - 1)$ = $(n - log_2(n+1))$. Therefore, finally, $h_{diff_{max}} - h_{diff_{min}}$ = $(n - log_2(n+1))$ - $((n-1) - log_2 n)$ = $(n - n + 1 - log_2(n+1) + log_2 n)$ = $(log_2(n) - log_2(n+1) + log_2 2)$ = $log_2(2n) - log_2(n+1)$ = $log_2 \frac{2n}{n+1}$ = $log_2(2 - \frac{2}{(n+1)})$. □

The difference, (n_{max} - n_{min}) defines the maximum deviation in the number of records with an actual height, h. Hence, n_{mdev} = n_{max} - n_{min} = ($2^{(h+1)}$−1 - $(h+1)$) = ($2^{(h+1)} - h - 2$).

The deviation in height, h_{dev} is defined as the deviation of the actual height, h from the optimal height, h_o. This is expressed as % of h_o. Mathematically:
h_{dev} = $\frac{h - h_o}{h_o}$ $\times 100\%$.



h_a & h_o h_a - h_o plot

Fig. 1. The actual and the optimal heights of the generated BSTs and their differences are plotted against the number of records, n. (The lower curve in Fig. (a) represents the optimal height).

The plot in Fig. $1(a)$ shows the height deviation of the actually generated BST from the optimal one. For the corresponding optimal BSTs, the height does not change from $n = 600$ to $n = 1000$. If $h_{opt} = 8$, the maximum number of records that it may contain is, $2^{8+1} - 1 = 511$. Whereas, if $h_{opt} = 9$, the maximum number of records it may contain is = $2^{9+1} - 1 = 1,023$. Therefore, for any value of n ranging from 600 to $1,000$, the optimal height is 9.

The sparsity factor is defined as, $sf = \frac{n_{max} - n}{n_{max}} \times 100\%$. Therefore, sf is required to be as small as possible. Since, n_{max} will be fixed for a particular value of h, the smaller the value of $(n_{max} - n)$, n will be closer to n_{max}, and the tree will approach that of the optimal configuration. This means that the sparsity will decrease. Though sparsity factor sf is suppose to decrease with the increasing value of n, but almost constant sf is an indicator of the relatively steady BST structures. Constant values of sf indicate that the actual number of nodes, n is relatively steady in comparison to the exponential growth of, $n_{max} = (2^{(h+1)} - 1)$ with the changing values of h.

The density factor, df = $\frac{(n - n_{min})}{n_{min}} \times 100\%$. For a constant value of h, the higher the density factor, df, n will become relatively larger and larger in comparison to n_{min}, and the tree will grow relatively denser.

IV. Multi-key BST Search Algorithm

Using Multi-key Search, it is possible to identify 1 or more subtrees in the original BST that starts at a particular record and ends at another one. Since such subtrees are just parts of the original BST, operations on these may be substantially faster than originally constructing those subtrees from the scratch. Using the proposed algorithm, it is possible at first to identify the subtree structure, and then applying the memory move operations, it is also possible to create a BST out of the subtree for further consideration.

```
  Algorithm find_record
  Purpose: This algorithm finds a record in the generated BST.
Require: name_supplied and this_node as inputs.
  if name_supplied.compareTo(this_node.name)== $0$ then
    return this_node
  else if name_supplied.compareTo(this_node.name) $< 0$ then
    if this_node.getLeftChild() is not NULL then
      return find_record (name_supplied, this_node.getLeftChild()) {recursive call to find_record}
    else
      return NULL
    end if
  else
    if this_node.getRightChild() is not NULL then
      return find_record (name_supplied, this_node.getRightChild())
    else
      return NULL
    end if
  end if
```

The 2-key binary search tree search algorithm makes use of the classical 1-key version.

```
  Algorithm find_record_2key
  Purpose: This algorithm performs 2-key binary search tree search.
  The supplied parameters are: array names[], current node verified this_node.
  find_record_2key finds out two matching nodes if available for the array names[] and return those as array search2[].
Require: names[0].compareTo(names[1]) $< 0$
```

```
Ensure: an array of correct records or NULLs are returned
  if names[1].compareTo(this_node.name) $< 0$ then
    if this_node.getLeftChild() is not NULL then
      search2[0] $\Leftarrow$ find_record (names[0], this_node.getLeftChild())
      search2[1] $\Leftarrow$ find_record (names[1], this_node.getLeftChild()) {Make 2 calls to find_record on the left subtree}
    else
      search2[0] $\Leftarrow$ NULL
      search2[1] $\Leftarrow$ NULL
    end if
    return search2[]
  else if names[0].compareTo(this_node.name) $> 0$ then
    if this_node.getRightChild() is not NULL then
      search2[0] $\Leftarrow$ find_record (names[0], this_node.getRightChild())
      search2[1] $\Leftarrow$ find_record (names[1], this_node.getRightChild()) {Make 2 calls to find_record on the right subtree}
    else
      search2[0] $\Leftarrow$ NULL
      search2[1] $\Leftarrow$ NULL
    end if
    return search2[]
  else if names[0].compareTo(this_node.name) $< 0$ and names[1].compareTo(this_node.name) $> 0$ then
    if this_node.getRightChild() is not NULL and this_node.getLeftChild() is not NULL then
      search2[0] $\Leftarrow$ find_record (names[0], this_node.getLeftChild())
      search2[1] $\Leftarrow$ find_record (names[1], this_node.getRightChild()) {Make 2 calls to find_record on two subtrees}
    else if this_node.getRightChild() is not NULL then
      search2[0] $\Leftarrow$ NULL
      search2[1] $\Leftarrow$ find_record (names[1], this_node.getRightChild())
    else if this_node.getLeftChild() is not NULL then
      search2[0] $\Leftarrow$ find_record (names[0], this_node.getLeftChild())
      search2[1] $\Leftarrow$ NULL
    else
      search2[0] $\Leftarrow$ NULL
      search2[1] $\Leftarrow$ NULL
    end if
    return search2[]
  else if names[0].compareTo(this_node.name) $== 0$ then
    if this_node.getRightChild() is not NULL then
      search2[0] $\Leftarrow$ this_node
      search2[1] $\Leftarrow$ find_record (names[1], this_node.getRightChild())
    else
      search2[0] $\Leftarrow$ this_node
      search2[1] $\Leftarrow$ NULL
    end if
    return search2[]
  else if names[1].compareTo(this_node.name) $== 0$ then
    if this_node.getLeftChild() is not NULL then
      search2[1] $\Leftarrow$ this_node
      search2[0] $\Leftarrow$ find_record (names[1], this_node.getLeftChild())
    else
      search2[1] $\Leftarrow$ this_node
      search2[0] $\Leftarrow$ NULL
    end if
    return search2[]
  end if
  return search2[]
```

A. Multi-key BST Search Analysis

For clarity, consider the 2-key BST search in this analysis. Suppose that the 1-st key is located at height h_1 and at k_1 position counting from the left-most record at height h_1. Similarly, suppose that the 2nd key is at height h_2 and at k_2 position counting from the left-most record at height h_2. Following are the possible scenarios with this 2-key BST Search.

- The 1st key is on the left subtree, and the 2nd key is on the right subtree of the root record. In this case, the subtree starting at k_1 and ending at k_2 includes the root node.
- Both k_1 and k_2 are on the left subtree. Then the subtree starting at k_1 and ending at k_2 will not include the root, and contains only a part of the left subtree.
- Both k_1 and k_2 are on the right subtree. Then the subtree starting at k_1 and ending at k_2 will not include the root, and contains a part of the right subtree.
- k_1 is the root node and k_2 is on right subtree. The subtree spanning from k_1 to k_2 includes a portion of the right subtree including the root.
- k_1 is on the left subtree and k_2 is at the root. In this case, the subtree from k_1 to k_2 contains a portion of the left subtree, which includes the root.

Following analysis is based on the assumption that the BST is complete up to a height of maximum$\{h_1, h_2\}$. Suppose that the height of the BST is h and the total number of records is n. Therefore, $h_1 \leq h$, and $h_2 \leq h$. For the following analysis, the left and the right subtrees counting from the root record are considered as two separate subtrees, since all calculations begin at the root and proceed either through the left subtree or along the right subtree of the root.

- Suppose $h_1 < h_2$, and both key_1 and key_2 are on the left subtree. Then for key_1, it is complete up to the level $(h_1 - 1)$, and there are k_1 records at level h_1 counting from the left-most node at the same height. Similarly, for key_2, it is complete up to the level $(h_2 - 1)$, and there are k_2 records at level h_2 counting from the left-most node. Since, $h_1 < h_2$, therefore, the total number of nodes in between key_1 and key_2, which are on the left subtree is, $n_{12} = \frac{1}{2}(2^0 + 2^1 + \ldots + 2^{h_2-1}) + k_2 - \frac{1}{2}(2^0 + 2^1 + \ldots + 2^{h_1-1}) - k_1 = \frac{1}{2}(2^{h_1} + 2^{h_1+1} + 2^{h_1+2} + \ldots + 2^{h_2-1}) + (k_2 - k_1) = (2^{h_2-1} - \frac{1}{2} - 2^{h_1-1} + \frac{1}{2} + k_2 - k_1) = 2^{h_1-1}(2^{h_2-h_1} - 1) + (k_2 - k_1)$.

- Let $h_1 = h_2$, and both key_1 and key_2 are on the left subtree. Since the assumption is that the keys are organized in ascending order and therefore, $k_1 < k_2$. Hence, n_{12} = $(k_2 - k_1)$.
- Suppose that $h_1 < h_2$, and k_1 is on the left subtree and k_2 is on the right subtree. In this case, n_{12} = $\frac{1}{2}(2^0 + 2^1 + \ldots + 2^{h_1 - 1} + k_1)$ + $\frac{1}{2}(2^0 + 2^1 + 2^2 + \ldots + 2^{h_1} + 2^{h_1+1} + \ldots + 2^{h_2 - 1})$ + $(k_2 - \frac{1}{2} \times 2^{h_2})$. Since the tree is complete up to the level 2^{h_2}, therefore $\frac{1}{2}$ of the total nodes up to the h_2 level lies on the left subtree, and the rest $\frac{1}{2}$ are on the right subtree, and k_1 and k_2 are counted starting from the left-most node on the left subtree. Hence, n_{12} = $\frac{1}{2}(2^{h_1} - 1)$ + $\frac{1}{2}\times$ $(2^{h_2} - 1)$ + $k_1 + k_2$ - $\frac{1}{2}\times$ (2^{h_2}) = $(\frac{1}{2} \times 2^{h_1} - 1 + k_1 + k_2)$ = $(2^{h_1 - 1} - 1 + k_1 + k_2)$.
- Suppose that $h_1 > h_2$, and k_1 is on the left subtree and k_2 is on the right subtree. In this case, n_{12} = $\frac{1}{2}(2^{h_1} - 1)$ + $\frac{1}{2}\times$ $(2^{h_2} - 1)$ + $k_1 + k_2$ - $\frac{1}{2}\times$ (2^{h_2}) = $(\frac{1}{2} \times 2^{h_1} - 1 + k_1 + k_2)$ = $(2^{h_1 - 1} - 1 + k_1 + k_2)$.
- Suppose that h_1 is equal to h_2, and k_1 is on the left subtree, and k_2 is on the right subtree. Here, n_{12} = $\frac{1}{2}(2^{h_1} - 1)$ + $\frac{1}{2}\times$ $(2^{h_1} - 1)$ + $k_1 + k_2$ - $\frac{1}{2}\times$ (2^{h_1}) = $(\frac{1}{2} \times 2^{h_1} - 1 + k_1 + k_2)$ = $(2^{h_1 - 1} - 1 + k_1 + k_2)$.
- $h_1 < h_2$, and both key_1 and key_2 are on the right subtree. For key_1, it is complete up to the level $(h_1 - 1)$, and there are k_1 records at level h_1 counting from the left. Similarly, for key_2, it is complete up to the level $(h_2 - 1)$, and there are k_2 records at level h_2 counting from the left. Since $h_1 < h_2$, therefore, the total number of records in between key_1 and key_2, which are on the right subtree is, n_{12} = $\frac{1}{2}(2^0 + 2^1 + \ldots + 2^{h_2 - 1}) + (k_2 - \frac{1}{2}(2^{h_2})) - \frac{1}{2}(2^0 + 2^1 + \ldots + 2^{h_1 - 1}) - (k_1 - \frac{1}{2}(2^{h_1}))$ = $(2^{h_2 - 1} - \frac{1}{2} + \frac{1}{2} - 2^{h_1 - 1} + 2^{h_2 - 1} - 2^{h_1 - 1} + k_2 - k_1)$ = $(2^{h_2} - 2^{h_1}) + (k_2 - k_1)$.
- $h_1 = h_2$, and both key_1 and key_2 are on the right subtree. Since the assumption is that the keys are organized in ascending order, therefore, $k_1 < k_2$. Hence, n_{12} = $k_2 - \frac{1}{2}(2^{h_1}) - (k_1 - \frac{1}{2}(2^{h_1}))$ = $k_2 - k_1$.
- key_1 is at the root and key_2 is on the right subtree. In that event, key_1 is at level 0 and the only possible record. Assuming key_2 is at level h_2, and the BST is complete up to the level h_2, n_{12} = $1 + \frac{1}{2}(2^1 + 2^2 + \ldots + 2^{h_2 - 1}) + (k_2 - \frac{1}{2}(2^{h_2})))$ (inclusive) = $1 + \frac{1}{2}(2^{h_2} - 1) - \frac{1}{2} + k_2 - \frac{1}{2}(2^{h_2})$ = k_2.
- key_1 is on the left subtree and key_2 is the root node. In that event, key_2 is at level 0 and the only possible record. Assuming key_1 is at level h_1, and the BST is complete up to the level h_1, n_{12} = $1 + \frac{1}{2}(2^1 + 2^2 + \ldots + 2^{h_1 - 1}) + k_1$ = $1 + k_1 + \frac{1}{2}(2^{h_1} - 1) - \frac{1}{2}$ = $k_1 + 2^{h_1 - 1}$.

Following the analysis of 2-key BST search, for the 3-key BST search, following are the possible scenario:

- The first 2 keys k_1 and k_2 exist on the left subtree, and the 3rd key, k_3 exists on the right subtree. In that event, the subtree starting at k_1 and ending at k_3 (inclusive) includes the root node.
- The 1st key, k_1 exists on the left subtree, and the 2nd and the 3rd keys, k_2 and k_3, respectively, are on the right subtree. In that case, the subtree starting at k_1 and ending at k_2 (inclusive) includes the root, whereas the subtree starting at k_2 and ending at k_3 includes only a part of the right subtree.
- All 3 keys, k_1, k_2 and k_3 exist on the right subtree. In that event, the subtree starting at k_1 and ending at k_3 (inclusive) does not include the root.
- All 3 keys k_1, k_2 and k_3 exist on the left subtree. In this case also, the subtree starting at k_1 and ending at k_3 does not include the root.
- k_1 is on the left subtree, k_2 is at the root and k_3 is on the right subtree. In this case, the subtree starting at k_1 and ending at k_3 includes the root (inclusive).
- k_1 is at the root, k_2 and k_3 are on the right subtree. In this case also, the subtree starting at k_1 and ending at k_3 (inclusive) includes the root and a part of the right subtree only.
- k_1 and k_2 are on the left subtree and k_3 is at the root. In that case, the subtree from k_1 to k_3 (inclusive) includes the root node.

B. Special Considerations

- 1st key is on the left subtree at level $h_1 \leq h$, and the 2nd key is at level $h_2 \leq h$ on the right subtree counting from the root node. In this case, the subtree starting at k_1 and ending at k_2 includes the root. If k_1 is the left-most record and k_2 is the right-most record, then the total number of records starting at k_1 and ending at k_2, n_{12} = $\frac{1}{2}(2^0 + 2^1 + 2^2 + \ldots + 2^{h_1}) + \frac{1}{2}(2^0 + 2^1 + 2^2 + \ldots + 2^{h_2})$ = $\frac{1}{2} \times \frac{(2^{h_1+1} - 1)}{(2-1)} + \frac{1}{2} \times \frac{(2^{h_2+1} - 1)}{2-1})$ = $\frac{1}{2}(2^{h_1+1} - 1) + \frac{1}{2}(2^{h_2+1} - 1)$ = $\frac{1}{2}(2^{h_1+1} + 2^{h_2+1} - 2)$ (since from the assumption, all the levels up to h_1 and h_2 are complete on the left and the right subtrees, respectively) = $\frac{1}{2}(2 \times 2^{h_1} + 2 \times 2^{h_2} - 2)$ = $\lfloor 2^{h_1} + 2^{h_2} - 1 \rfloor$ = Maximum number of records possible between k_1 and k_2 (inclusive).
- If k_1 is the right-most record on the left subtree at level h_1, and k_2 is the left-most record on the right subtree at level h_2. Assuming that all levels up to $(h_1 - 1)$ are complete on the left subtree, and all the levels up to $(h_2 - 1)$ are complete on the right subtree as well. Then n_{12} = $\frac{1}{2} \times (2^0 + 2^1 + 2^2 + \ldots + 2^{h_1 - 1}) + 1) + \frac{1}{2} \times (2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^{h_2 - 1}) + 1)$ = $\frac{1}{2} \times \frac{(2^{h_1 - 1 + 1} - 1)}{2-1} + 1 + \frac{1}{2} \times \frac{(2^{h_2 - 1 + 1} - 1)}{(2-1)} + 1$ = $\frac{1}{2} \times 2^{h_1} - \frac{1}{2} + 1 + \frac{1}{2} \times 2^{h_2} - \frac{1}{2} + 1$ = $2^{h_1 - 1} + 2^{h_2 - 1} + 1$ = $\frac{1}{2}(2^{h_1} + 2^{h_2} + 2)$ = Minimum number of nodes (inclusive) possible between k_1 and k_2.
- Therefore, following relationship holds true for k_1 in the left subtree and k_2 on the right one: $\lfloor \frac{1}{2}(2^{h_1} + 2^{h_2} + 2) \rfloor \leq n_{12} \leq \lfloor (2^{h_1} + 2^{h_2} - 1) \rfloor$. Hence, $(n_{12_{max}} - n_{12_{min}})$ = $(2^{h_1} + 2^{h_2} - 1) - \frac{1}{2}(2^{h_1} + 2^{h_2} + 2)$ = $2^{h_1 - 1} + 2^{h_2 - 1} - 2$.
- Let k_1 is the left-most node at level h_1 in the left subtree and k_2 is the right-most node at level h_2 in the left subtree as well. Assuming all the intermediate levels including the levels at h_1 and h_2 are complete, the total number of nodes (inclusive), n_{12} = $\frac{1}{2}(2^{h_1} + \ldots + 2^{h_2})$ = $\frac{1}{2}(2^0 + 2^1 + \ldots + 2^{h_1 - 1} + 2^{h_1} + \ldots + 2^{h_2})$ - $\frac{1}{2}(1 + 2^1 + \ldots + 2^{h_1 - 1})$ = $\frac{1}{2}(\frac{(2^{h_2+1} - 1)}{(2-1)}) - \frac{1}{2}(\frac{(2^{h_1} - 1)}{(2-1)}$ = $2^{h_2} - \frac{1}{2} - 2^{h_1 - 1} + \frac{1}{2}$ = $(2^{h_2} - 2^{h_1 - 1})$, which gives us the expression for the maximum number of records on the same subtree. Hence,

$n_{12} \leq (2^{h_2} - 2^{h_1 - 1})$.

- Let k_1 be the right-most node at level h_1, and k_2 is the left-most node at level h_2 on the same subtree. Then $h_1 < h_2$. Hence, the minimum number of nodes in between k_1 and k_2 (inclusive) is = $1 + \frac{1}{2} \times (2^{h_1+1} + \ldots + 2^{h_2-1}) + 1$ = 2 + $\frac{1}{2}(2^0 + 2^1 + \ldots + 2^{h_1} + 2^{h_1+1} + \ldots + 2^{h_2-1})$ - $\frac{1}{2}(2^0 + 2^1 + \ldots + 2^{h_1})$ = 2 + $\frac{1}{2}(\frac{2^{h_2-1+1}-1}{(2-1)}) - \frac{1}{2}(\frac{(2^{h_1+1}-1)}{(2-1)}$ = 2 + $\frac{1}{2}(2^{h_2} - 1) - \frac{1}{2}(2^{h_1+1} - 1)$ = $(2 + 2^{h_2-1} - 2^{h_1})$. Therefore, $n_{12} \geq (2 + 2^{h_2-1} - 2^{h_1})$.
- If both the keys are on the same subtree, then the following relation holds true: $(2 + 2^{h_2-1} - 2^{h_1}) \leq n_{12} \leq (2^{h_2} - 2^{h_1-1})$. Hence, $(n_{12_{max}} - n_{12_{min}})$ = $(2^{h_2} - 2^{h_1-1}) - (2 + 2^{h_2-1} - 2^{h_1})$ = $2^{h_2-1} + 2^{h_1-1} - 2$.
- If key_1 is at the root, and key_2 is on the right subtree, then k_1 is at level 0, and the only possible node. Assuming k_2 is at level h_2, there are 2 instances possible. In one instance, k_2 is the right-most node of the right subtree at level h_2. In that event, $n_{12_{max}}$ = $1 + \frac{1}{2}(2^1 + 2^2 + \ldots + 2^{h_2})$ (inclusive) = $1 - \frac{1}{2} + \frac{1}{2}(2^{h_2+1} - 1)$ = 2^{h_2}. On the other hand, if k_2 is the left-most node, and the only record available at level h_2, $n_{12_{min}}$ = $1 + \frac{1}{2}(2^1 + 2^2 + \ldots + 2^{h_2-1}) + 1 = 2 + \frac{1}{2}(2^{h_2} - 1) - \frac{1}{2}$ = $2 - \frac{1}{2} - \frac{1}{2} + 2^{h_2-1}$ = $1 + 2^{h_2-1}$. With k_1 at the root, and k_2 on the right subtree, following is the restriction: $1 + 2^{h_2-1} \leq n_{12} \leq 2^{h_2}$. Also, $(n_{12_{max}} - n_{12_{min}})$ = $(2^{h_2} - 1 - 2^{h_2-1})$ = $(2^{h_2-1} - 1)$.
- If key_1 is on the left subtree, and key_2 is at the root, then k_2 is at level 0, and the only possible node. Hence, $n_{12_{max}}$ = $1 + \frac{1}{2}(2^1 + 2^2 + \ldots + 2^{h_1})$ (inclusive) = $1 + \frac{1}{2}(2^{h_1+1} - 1) - \frac{1}{2}$ = $1 + 2^{h_1} - \frac{1}{2} - \frac{1}{2}$ = 2^{h_1}. Once again, $n_{12_{min}}$ = $1 + \frac{1}{2}(2^1 + 2^2 + \ldots + 2^{h_1-1}) + 1$ (inclusive) = $2 + \frac{1}{2}(2^{h_1} - 1) - \frac{1}{2}$ = $1 + 2^{h_1-1}$. Therefore, $(n_{12_{max}} - n_{12_{min}})$ = $2^{h_1} - 1 - 2^{h_1-1}$ = $2 \times 2^{h_1-1} - 1 - 2^{h_1-1}$ = $2^{h_1-1} - 1$.
- Next consider a very special case of pruning the right subtree of the 1st key and the left subtree of the 2nd key in the two key BST Search together with the other nodes from key_1 to key_2. Further assume that key_1 is on the left subtree and key_2 is on the right subtree. If key_1 is at level h_1 and key_2 is at level h_2 with $h_1 \leq h$ and $h_2 \leq h$, where h is height of the BST, then the number of nodes in the right subtree of the 1st key = $2^0 + 2^1 + \ldots + 2^{(h-h_1-1)}$ = $\frac{2^{(h-h_1-1+1)}-1}{2-1}$ = 2^{h-h_1}. Similarly, the total number of nodes in the left subtree of the 2nd key = $2^0 + 2^1 + \ldots + 2^{(h-h_2-1)}$ = $\frac{2^{(h-h_2-1+1)}-1}{2-1}$ = 2^{h-h_2}. These numbers may be computed by imagining a BST subtree with root at the right child of the 1st key and another subtree having the root at the left child of the 2nd key. Off-course for this analysis to hold true, the BST is required to be complete.

  If $h_1 < h_2$, and k_1 is on the left subtree and k_2 is on the right subtree, n_{12} = $(2^{h_1-1} - 1 + k_1 + k_2)$. Hence, total number of nodes in the pruned subtree = n_{12} + 2^{h-h_1} + 2^{h-h_2} = $(2^{h_1-1} - 1 + k_1 + k_2)$ + 2^{h-h_1} + 2^{h-h_2}

  Similarly, if $h_1 > h_2$, total number of nodes in the pruned subtree = $(2^{h_1-1} - 1 + k_1 + k_2)$ + 2^{h-h_1} + 2^{h-h_2}.

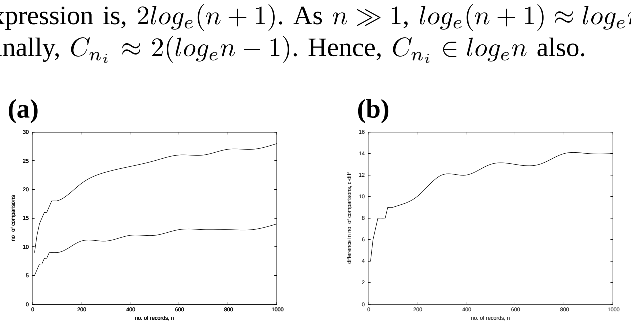
  Also if $h_1 = h_2$, total number of nodes in the pruned subtree = $(2^{h_1-1} - 1 + k_1 + k_2)$ + 2^{h-h_1} + 2^{h-h_2}.

V. Multi-Key Search Performance

In this section, the results pertaining to multiple key-based searches in BSTs are considered.

Theorem 2: The average cost of an unsuccessful search is given by, C_{n_e} = $2log_e(n + 1)$, and the average cost C_{n_i} for a successful search is = 2($log_e n$ -1).

Proof: Let E_n be the expected external path length, and I_n be the expected internal path length for a BST with n nodes. Therefore, following recurrence relation holds true: $E_o = 0$, $E_n = (n + 1) + \frac{1}{n}\sum_{i=1}^{n}(E_{i-1} + E_{n-i})$. Here, $(n + 1)$ is to account for the root node cost for each external node (there are $(n + 1)$ of them). $\frac{1}{n}\sum_{i=1}^{n}(E_{i-1} + E_{n-i})$ accounts for the average external path length of the left and the right subtrees over with an arbitrary ith element at the root. But $\sum_{i=1}^{n} E_{i-1} = \sum_{i=1}^{n} E_{n-i})$. Hence, rewriting the expression for E_n, E_n = $(n+1) + \frac{2}{n}\sum_{i=0}^{n-1} E_i$. Then $E_{n-1} = n + \frac{2}{n-1}\sum i = 0^{n-2}E_i$, which provides: $(n-1)E_{n-1} - n(n-1) = 2\sum_{i=0}^{n-2} E_i$. Now expanding the expression for E_n, $E_n = (n + 1) + \frac{2}{n}E_{n-1} + \frac{2}{n}\sum_{i=0}^{n-2} E_i$. From the previous expression for E_{n-1}, $2\sum_{i=0}^{n-2} E_i = E_{n-1} - n$. Hence, $E_n = (n+1) + \frac{2}{n}E_{n-1} + \frac{1}{n}[(n-1)E_{n-1} - n(n-1)]$. From this last expression, $E_n = 2 + \frac{(n+1)}{n}E_{n-1}$. Expanding the recurrence relationship for E_n, E_n = $2 + 2\frac{(n+1)}{n} + \frac{(n+1)}{n} \times \frac{n}{(n-1)}E_{n-2}$ = $2 + 2\frac{(n+1)}{n} + \frac{(n+1)}{(n-1)}E_{n-2}$ = $2 + 2\frac{(n+1)}{n} + \frac{(n+1)}{(n-1)} + \frac{(n+1)}{(n-2)}E_{n-3}$ = $2 + 2\sum_{i=1}^{n}\frac{(n+1)}{i} + (n+1)E_0 = 2 + 2\sum_{i=1}^{(n+1)}\frac{(n+1)}{i} - 2\frac{(n+1)}{(n+1)}$ = $2(n+1)\sum_{i=1}^{(n+1)}\frac{1}{i}$ (as E_0 = 0) = $2(n+1)H_{n+1}$. Here, H_{n+1} is the $(n + 1)$th Harmonic Function. Using the properties of Harmonic Functions, $H_{n+1} \approx log_e(n + 1) + \gamma$, where γ = Euler's constant ≈ 0.5772. Hence, $E_n \approx 2(n + 1)(log_e(n + 1) + \gamma)$. The average cost for an unsuccessful search is, $\frac{E_n}{(n+1)}$ = $2(log_e(n + 1) + \gamma) \approx 2log_e(n + 1)$ (see [1]). The highest order term in this expression is: $2log_e(n + 1)$. Therefore, $C_{n_e} \in log_e n$. Again, $E_n = I_n + 2n$. This provides, $I_n = 2(n+1)(log_e(n+1) + \gamma) - 2n$. The average cost of a successful search is, $\frac{I_n}{n} \approx \frac{2(n+1)(log_e(n+1)-2n}{n} \approx 2(1 + \frac{1}{n})log_e(n+1) - 2$ $\approx (2log_e(n + 1) + \frac{2}{n}log_e(n + 1) - 2)$. As $n \gg 1$, $\frac{2}{n} \approx 0$. Hence, $C_{n_i} \approx 2log_e(n+1) - 2$. The highest order term in this expression is, $2log_e(n + 1)$. As $n \gg 1$, $log_e(n + 1) \approx log_e n$. Finally, $C_{n_i} \approx 2(log_e n - 1)$. Hence, $C_{n_i} \in log_e n$ also. □



$S(n)$ & $U(n)$ curves $U(n) - S(n)$ plot

Fig. 2. Average number of comparisons for successful ($S(n)$) and unsuccessful searches ($U(n)$) are plotted against the total number of records, n. (The lower curve in Fig. (a) represents the curve for successful searches).

From the standard DS literature, average number of comparisons for a successful search, $S(n)$ = $\lceil 1.39log_2 n \rceil$. Average number of comparisons for an unsuccessful search, $U(n)$ =

$\lceil 2.77 log_2 n \rceil \approx 2 \times S(n)$. Both $S(n)$ and $U(n)$ depends on $log_2 n$. The minimum value of n is, n_{min} = 10, and the maximum value of n is, n_{max} = $1,000$. For example, when $n = 10$, $S(n)$ = 4.62, and $U(n)$ = $9.2 \approx 2 \times S(n)$. Hence, the curve for average unsuccessful search, $U(n)$ follows almost exact pattern to that of the successful search, $S(n)$. Again, the difference between $U(n)$ and $S(n)$ is, $d(n)$ = $U(n) - S(n)$ = $\lceil 2.77 log_2 n \rceil$ - $\lceil 1.39 log_2 n \rceil$ = $1.38 log_2 n \approx S(n)$. Therefore, the difference curve, $d(n)$ in Fig. $2(b)$ has almost exactly the same pattern as that of $S(n)$.

VI. Performance Issues in Practice

Lemma 3: An m-key binary search tree search algorithm may be applied to any BST containing n records, where $n >= m$.

Proof: A proof by contradiction is adopted. Suppose that $n < m$. Therefore, the total number of keys to search for in the BST becomes greater than the number of records within the BST. In the best possible case, n different keys may be identified at the n record positions, leaving (m-n) keys undecided during the computation, for which, no records to look for may be available. This violates the objective of the m-key BST search, which is to identify the BST records corresponding to the m-keys within the BST. Hence, $m \not> n$, and at most, m = n. □

Theorem 4: An m-key BST search requires considering ($2m$+1) different cases in identifying the records corresponding to the m keys inside the BST. Here, $m \geq 1$.

Proof: Following is a proof by mathematical induction.

Base Case: For the base case, m=1. For P(1), it is the classical, single key BST search. It considers 3-different cases. These are: (1) key_element = root value, (2) key_element > root value, and (3) key_element < root value. Hence, (2 × 1 + 1) = 3 different cases are being considered.

Induction: Suppose that the k-key search algorithm requires considering ($2k$+1) different cases. Here, $k \geq 1$. It is required to show that: [P(1) $\bigwedge$ $\forall$ P(k)]$\rightarrow$ P(k+1), which is proving that for (k+1) different keys, (2(k+1) + 1) = $2k$+3 different cases are required to be considered. For the $(k+1)$th key, two more cases are required in addition to the ($2k+1$) cases for the first k keys. For the sorted keys, $(k + 1)$th key is the largest and the last key within the list. Therefore, it requires considering only 2 additional cases. First of all, verifying whether the root value is equal to the $(k + 1)$th key value. If so, the $(k + 1)$th key is found at the root, and it is necessary to use the k-key steps of the BST search to locate the first k-keys. Secondly, it is required to verify whether the $(k + 1)$th key is larger, and the kth key is smaller than the root node. In that event, confine search for the $(k+1)$th key to the right half of the BST using a classical BST search, and use the steps of the k-key BST search for the first k keys. Rest of the cases are identical to the k-key version except that it is required to consider (k+1) keys instead of k keys. Hence, altogether, for the (k+1) key version, we require considering ($2k + 1$ + 2) = $2(k + 1)$ + 1 different cases.

Conclusion: The theorem is true for $m = 1$. Assuming that the theorem holds true for $m = k$, it is proved that it also holds true for $m = (k + 1)$ different keys. As it is true for $m = 1$, it is also true for $m = 2$. As it holds true for $m = 2$, it is also true for $m = 3$, and so on. Hence, the theorem holds true for any m with $m \geq 1$. □

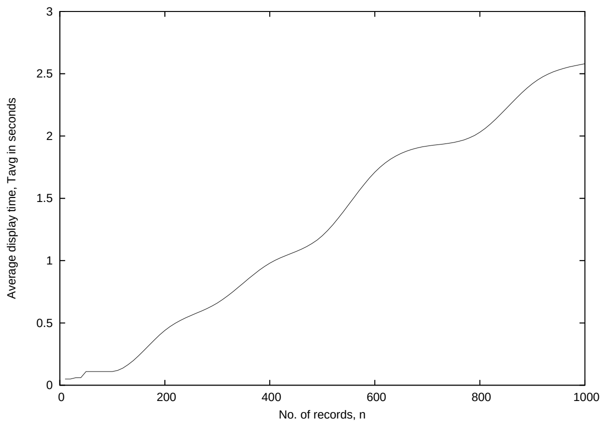


Fig. 3. Average time to display the sorted tree, T in seconds is plotted against the number of records, n inside the tree with data acquired using the Borland's C++ 5.02 compiler.

Fig. 3 shows time, T required to display the nodes in ascending order using in-order traversal of the BST, which is plotted against the size, n. The curve is obtained using Borland's C++ 5.02 compiler. The total time, T required to display the nodes contains two components. One is the fixed timing overhead, T_o required by the program to run on a platform. The other one is the total time, T_d to display the BST entries. Therefore, $T = T_o + T_d$. Initially, from $n = 10$ to $n = 100$, the time required to display the nodes T_d is insignificant compared to the timing overhead, T_o. Therefore, T_o dominates over T_d, as pronounced by the relatively flat nature of the curve within this range. For $n = 100$ to $n = 1,000$, T_d becomes relatively much higher compared to T_o, and T_d dominates over T_o. Therefore, within this range, the total time consumed depends more on T_d than only on T_o, which may be realized by the increasing nature of the curve.

Fig. 4 shows the 1-key, the 2-key and the 3-key BST inorder traversal times plotted against n for using the Java JDK 5.0 compiler running on Pentium 4 machine. The total time taken rapidly increases with the increasing number of keys as the time to display the combinations is approximately in the order of, $O(n^m)$. Here, n is the number of elements and m is the number of keys. As $n \gg m$, m remains relatively constant with respect to n, and the curves display polynomial characteristic as expected.

VII. Conclusion

In this paper, some new results on BST performance, and the deviation of the generated BST structures from the corresponding optimal structures are also presented. To search an array with speed, the array needs to be sorted. Inserting new nodes in an ordered array is a pain. Again, inserting new nodes in a linked list is easy; but searching for an existing node in a linked list is difficult. BSTs are data structures that combines the good properties of arrays and linked lists, but has

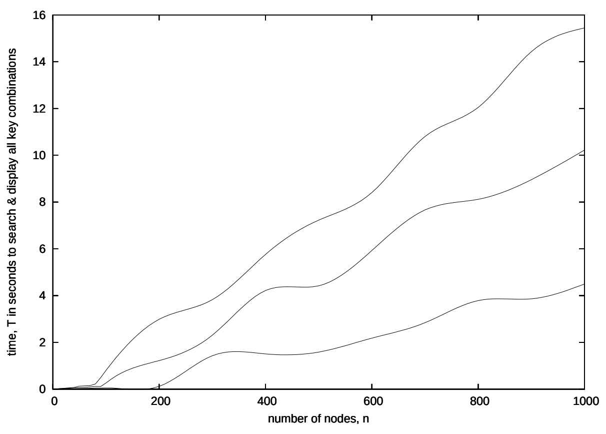


Fig. 4. Total time T_{total} to search and display all possible key combinations for 1-key, 2-key, and 3-key BST searches are plotted against the total number of nodes, n. Topmost curve is for the 3-keys considered together. The middle one is for possible 2-key combinations, and the lower most curve is for the casual 1-key BST search. Sun Java's JDK 5 compiler was used in performance measurement.

none of the bad parts. Hence, complete and nearly complete BSTs draw special attention to the data structure researchers.

Efficiency of a BST application depends on the depth of the tree, h. Therefore, a good number of the results are based on the height, h-based performance of the BSTs. In the best possible scenario, the optimal depth is, $h_{best} = log_2(n)$. If the keys are added at random to a gradually growing BST structure, it results in a BST with an average height, h_{avg}. For an average BST, $h_{avg} \approx 1.39log_2(n)$. Therefore, h_{avg} is only about 40% higher than the best possible.

In future, a dynamic programming model for generating an optimal BST structure with the minimal internal and the external path lengths will be developed, and the related performance issues will be addressed.

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