Mathematical model for small satellites, using attitude and rotation angles

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Abstract - The paper purpose is to present some aspects regarding the calculus model and technical solutions for small satellites attitude control. Mathematical model is put in nonlinear and linear form. The linear form is used for attitude control system synthesis. The attitude control system obtained is used in nonlinear form in order to maintain desired attitude. A few numerical simulations are made for standard input and the satellite behavior is obtained. The satellite model presented will be with six DOF and uses Cartesian coordinates. At this item, as novelty of the work we will use the rotation angles to describe the kinematical equations. Also this paper proposes a Fourier linearising of *Trigger Schmidt* element used for applying the command moment. The results analyzed will be the rotation angles of the satellite as well the rotation velocity. The conclusions will focus the comparison between results obtained using different attitude control system, and the possibility to use such system for small satellite.

Keywords— Automatic, Attitude control system**,** Mathematic model, Simulation, Small satellite

NOMENCLATURE

ξ - Rotation angle around body X_B axis
η - Rotation angle around body Y_B axis
ζ - Rotation angle around body Z_B axis
ψ - Attitude angle around z axis
θ - Attitude angle around y axis
ϕ - Attitude angle around x axis
$\boldsymbol{\omega}_{BI}$ -Angular velocity of the body frame relative to the inertial frame expressed in body frame;
$\boldsymbol{\omega}_{RI}$ -Angular velocity of the reference frame relative to the inertial frame expressed in relative frame;
$\boldsymbol{\omega}_{RIB}$ -Angular velocity of the reference frame relative to the inertial frame expressed in body frame;

$\boldsymbol{\omega}_{BR}$ -Angular velocity of the body frame relative to the reference frame expressed in body frame;

A, B, C, E - Satellite inertia moments;
m - Satellite Mass;
a - Major semi axis of elliptical satellite orbit;
e - Eccentricity of elliptical satellite orbit;
t - Time;
$\mathbf{r}$ - Position vector of satellite relative to origin of the inertial frame – center of the Earth;
T -Orbital period;
$\mathbf{v}$ - Velocity;
V_{Ix}, V_{Iy}, V_{Iz} - Velocity components expressed in inertial frame;
X_B, Y_B, Z_B - Body frame;
X_R, Y_R, Z_R - Reference frame;
X_I, Y_I, Z_I -Inertial frame
x_I, y_I, z_I - Satellite coordinate expressed in inertial frame;

I. Introduction

It is indisputable that today, the use of satellites is the spatial program main goal, due to their importance in terms of telecommunications, remote sensing and navigation that they provide. During a satellite mission, the quality of Attitude Control System (ACS) is a basic element for achieving good functioning condition. For ACS completion is required to build a suitable mathematical model that allows both the synthesis of the control system and its simulation.

Starting from this requirement the paper will introduce two novelties:

-The use of the rotation angles for describing the kinematical equations;

-The linearising of *Trigger Schmidt* element used for applying the command moment, thru Fourier Transformation

Unlike regular paper, which covers the regular aircraft cases, where the kinematical equations use Euler angles, in our case, when the satellite has a complex evolution, the papers [4],[5] and [10] recommends the quaternion vector or the rotation angles. Using kinematical equations written with Euler angles, in addition to benefits related to the significance of physical measurable sizes, the following drawback is involved: the use of trigonometric functions in program algorithms. Although complications related to solve the kinematical

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equations, the rotation angles can be used for attitude control, as it will be shown next.

II. General movement equations

A. Used frames

First of all, we must define a frame, call referance frame (RF), in which the satellite will be stabilized related to three axes. As we can see in figure 1 the frame origin of referance frame is in the mass center of the satellite and moves with it. Axis $\mathbf{Z}_R$ is orientated towards Earth mass center, $\mathbf{X}_R$ axis is in orbit plane, normal of $\mathbf{Z}_R$ axis and orientated towards velocity direction. Axis $\mathbf{Y}_R$ is normal to the orbit plane, and completes an orthogonal right-hand system. In the same time we define an angular velocity $\boldsymbol{\omega}_{RI}$ which means angular velocity of the reference frame related to inertial frame. Inertial frame X_I, Y_I, Z_I (IF) has its origin in Earth center being use for the description of orbital moving of the satellite. The body frame $\mathbf{X}_B\mathbf{Y}_B\mathbf{Z}_B$ (BF) is an orthogonal frame heaving the axis, if is possible, along the principal inertial axis. The attitude of the boy frame related to the reference frame is defined using attitude angles type Euler, or quaternion vector or, as we will present later on, the rotation angles.

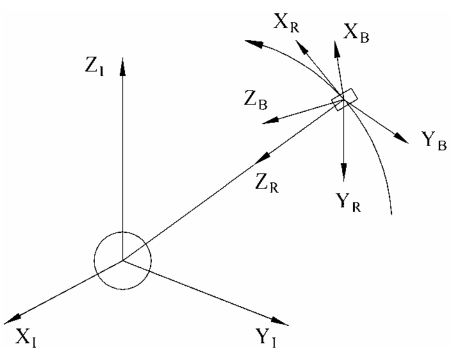

Fig. 1 definition of the used frames

B. The angular velocity of the body frame

The two main terms of the kinematical orientation of the satellite are:

- The angular velocity of the BF related the RF:

$$\boldsymbol{\omega}_{BR} = p_B\mathbf{i} + q_B\mathbf{j} + r_B\mathbf{k} \quad (1)$$

- The angular velocity of the RF related IF:

$$\boldsymbol{\omega}_{RI} = \omega_i\mathbf{i}_R + \omega_j\mathbf{j}_R + \omega_k\mathbf{k}_R \quad (2)$$

The angular velocity of the RF related IF $\boldsymbol{\omega}_{RI}$: expressed in BF is:

$$\boldsymbol{\omega}_{RIB} = \omega_{iB}\mathbf{i} + \omega_{jB}\mathbf{j} + \omega_{kB}\mathbf{k} \quad (3)$$

The link between this two expressions being on the rotation matrix:

$$\boldsymbol{\omega}_{RIB} = \mathbf{A}_e\boldsymbol{\omega}_{RI}, \quad (4)$$

where, the rotation matrix $\mathbf{A}_e$ will be further defined

Finally we define the angular velocity of the BF related to the IF :

$$\boldsymbol{\omega}_{BI} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k} \quad (5)$$

Between these three sizes there is the link:

$$\boldsymbol{\omega}_{BI} = \boldsymbol{\omega}_{BR} + \boldsymbol{\omega}_{RIB} \quad (6)$$

Because all components are expressed in the body frame, it can be write immediately the scalar link:

$$\begin{bmatrix}\omega_x & \omega_y & \omega_z\end{bmatrix}^T = \begin{bmatrix}p_B & q_B & r_B\end{bmatrix}^T + \mathbf{A}_e\begin{bmatrix}\omega_i & \omega_j & \omega_k\end{bmatrix}^T \quad (7)$$

Because the RF are defined by unitary vectors: $\mathbf{i}_R, \mathbf{j}_R, \mathbf{k}_R$, their definition based on position vector $\mathbf{r}$ and velocity $\mathbf{v}$:

$$\mathbf{k}_R = -\frac{\mathbf{r}}{r}\ ;\ \mathbf{j}_R = \frac{\mathbf{v}\times\mathbf{r}}{|v\times r|};$$

$$\mathbf{i}_R = \mathbf{j}_R \times \mathbf{k}_R = \frac{\mathbf{r}\times(\mathbf{v}\times\mathbf{r})}{r|\mathbf{v}\times\mathbf{r}|} = \frac{r^2\mathbf{v} - \mathbf{r}\cdot(\mathbf{v}\cdot\mathbf{r})}{r|\mathbf{v}\times\mathbf{r}|} \quad (1)$$

Deriving in reference frame we obtain successive:

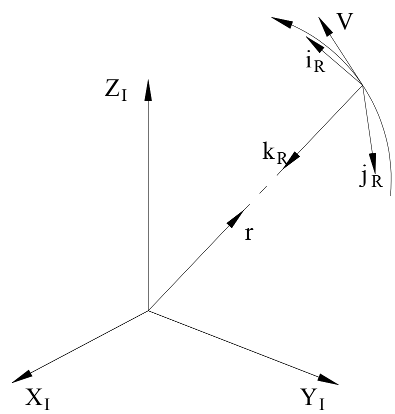

Fig. 2 definition of the unitary vectors for the reference frame

$$\frac{d\mathbf{i}_R}{dt} = (\omega_i\mathbf{i}_R + \omega_j\mathbf{j}_R + \omega_k\mathbf{k}_R)\times\mathbf{i}_R = \omega_k\mathbf{j}_R - \omega_j\mathbf{k}_R;$$

$$\frac{dj_R}{dt} = (\omega_i\mathbf{i}_R + \omega_j\mathbf{j}_R + \omega_k\mathbf{k}_R)\times j_R = \omega_i\mathbf{k}_R - \omega_k\mathbf{i}_R;$$

$$\frac{d\mathbf{k}_R}{dt} = (\omega_i\mathbf{i}_R + \omega_j\mathbf{j}_R + \omega_k\mathbf{k}_R)\times\mathbf{k}_R = \omega_j\mathbf{i}_R - \omega_i\mathbf{j}_R. \quad (10)$$

If we perform scalar product of each relation with unitary vectors $\mathbf{i}_R, \mathbf{j}_R, \mathbf{k}_R$, found that:

$$\omega_i = \frac{d\mathbf{j}_R}{dt}\mathbf{k}_R = -\frac{d\mathbf{k}_R}{dt}\mathbf{j}_R;\ \omega_j = \frac{d\mathbf{k}_R}{dt}\mathbf{i}_R = -\frac{d\mathbf{i}_R}{dt}\mathbf{k}_R;$$

$$\omega_k = \frac{d\mathbf{i}_R}{dt}\mathbf{j}_R = -\frac{d\mathbf{j}_R}{dt}\mathbf{i}_R \quad (11)$$

Taking into account by definition relations (8) and the features of mixed product we obtain successively:

$$\omega_i = -\frac{d\mathbf{k}_R}{dt}\mathbf{j}_R = \left(\frac{1}{r}\frac{d\mathbf{r}}{dt} - \frac{\dot{r}}{r^2}\mathbf{r}\right)\frac{\mathbf{v}\times\mathbf{r}}{|\mathbf{v}\times\mathbf{r}|} =$$

$$= \frac{1}{r|\mathbf{v}\times\mathbf{r}|}\mathbf{v}(\mathbf{v}\times\mathbf{r}) - \frac{\dot{r}}{r^2|\mathbf{v}\times\mathbf{r}|}\mathbf{r}(\mathbf{v}\times\mathbf{r}) \quad (12)$$

$$= \frac{1}{r|\mathbf{v}\times\mathbf{r}|}\mathbf{r}(\mathbf{v}\times\mathbf{v}) - \frac{\dot{r}}{r^2|\mathbf{v}\times\mathbf{r}|}\mathbf{v}(\mathbf{r}\times\mathbf{r}) = 0$$

Similarly, we obtain:

$$\omega_j = \frac{d\mathbf{k}_R}{dt}\mathbf{i}_R = \left(\frac{\dot{r}}{r^2}\mathbf{r} - \frac{1}{r}\mathbf{v}\right)\frac{r^2\mathbf{v} - \mathbf{r}\cdot(\mathbf{v}\cdot\mathbf{r})}{r|\mathbf{v}\times\mathbf{r}|} =$$

$$= \frac{1}{r^2|\mathbf{v}\times\mathbf{r}|}\left[(\mathbf{r}\cdot\mathbf{v})^2 - r^2v^2\right] \quad (13)$$

Taking into account that [14]:

$$|\mathbf{v}\times\mathbf{r}| = rv\cos\beta\,;\,\mathbf{r}\cdot\mathbf{v} = rv\sin\beta\,, \quad (14)$$

finally we obtain:

$$\omega_j = -\frac{v}{r}\cos\beta = -\frac{h}{r^2} \quad (14)$$

If the orbit is circular, obviously we have:

$$\omega_j = -\frac{v}{r} \quad (15)$$

Finally for obtain ω_k, develop such:

$$\omega_k = -\frac{dj_R}{dt}\mathbf{i}_R = -\frac{1}{|\mathbf{v}\times\mathbf{r}|}\frac{d}{dt}(\mathbf{v}\times\mathbf{r})\mathbf{i}_R = -\frac{1}{|\mathbf{v}\times\mathbf{r}|}[\ddot{\mathbf{r}}\times\mathbf{r} + \mathbf{v}\times\dot{\mathbf{r}}]\mathbf{i}_R =$$

$$= -\frac{1}{|\mathbf{v}\times\mathbf{r}|}(\ddot{\mathbf{r}}\times\mathbf{r})\mathbf{i}_R \quad (16)$$

But, for Keplerian case we can write:

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{r^3} \quad (17)$$

obtaining that:

$$\omega_k = \frac{\mu}{r^3|\mathbf{v}\times\mathbf{r}|}[\mathbf{r}\times\mathbf{r}]\mathbf{i}_R = 0 \quad (18)$$

Obviously, for Keplerian movement, no acceleration outside from the orbit plan, so we can write:

$$\omega_{RI} = \begin{bmatrix}0 & \omega_j & 0\end{bmatrix}^T \quad (19)$$

where:

$$\omega_j = -\frac{h}{r^2} \quad (20)$$

Taking into account of relation (4) finally we can write:

$$\begin{bmatrix}p_B & q_B & r_B\end{bmatrix}^T = \begin{bmatrix}\omega_x & \omega_y & \omega_z\end{bmatrix}^T - \mathbf{A}_e\begin{bmatrix}0 & \omega_j & 0\end{bmatrix}^T \quad (21)$$

This relationship is important because if we measure or calculate the angular velocity of the BF related to IF $\boldsymbol{\omega}_{BI} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$, then using (21) we can determine the angular velocity of the BF related with RF $\boldsymbol{\omega}_{BR} = \begin{bmatrix}p_B & q_B & r_B\end{bmatrix}^T$ and use it to control the vehicle attitude related to RF

C. Kinematical equations

Unlike paper [3], which covers the regular flight vehicles, where the kinematical equations use Euler angles, in our case, when we have unusual or unpredicted attitude, the papers [5], [10] recommend quaternion or rotation angles. Using kinematical equations written with Euler angles the following drawback is involved: singularity of the connection matrix and the use of trigonometric functions in program algorithms. To outline this disadvantage, further we will compare the attitude Euler angles with rotation angles.

For attitude angles case using partial rotation matrix:

$$\mathbf{A}_\psi = \begin{bmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\mathbf{A}_\theta = \begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}\mathbf{A}_\phi = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi & \sin\phi\\ 0 & -\sin\phi & \cos\phi\end{bmatrix}$$

the complete rotation matrix becomes:

$$\mathbf{A}_e = \mathbf{A}_{\phi,\psi,\theta} = \mathbf{A}_\phi\mathbf{A}_\psi\mathbf{A}_\theta, \quad (22)$$

If we denote:

$c\phi = \cos\phi$; $s\phi = \sin\phi$; $c\theta = \cos\theta$; $cs\theta = \sin\theta$; $c\psi = \cos\psi$; $s\psi = \sin\psi$

the rotation matrix is:

$$\mathbf{A}_e = \begin{bmatrix}c\psi c\theta & s\psi & -c\psi s\theta\\ -c\phi s\psi c\theta + s\phi s\theta & c\phi c\psi & c\phi s\psi s\theta + s\phi c\theta\\ s\phi s\psi c\theta + c\phi s\theta & -s\phi c\psi & -s\phi s\psi s\theta + c\phi c\theta\end{bmatrix} \quad (23)$$

The derivatives rotation matrix related the attitude angles are:

-The derivative related ϕ angle:

$$\frac{\partial\mathbf{A}_e}{\partial\phi} = \begin{bmatrix}0 & 0 & 0\\ s\phi s\psi c\theta + c\phi s\theta & -s\phi c\psi & -s\phi s\psi s\theta + c\phi c\theta\\ c\phi s\psi c\theta - s\phi s\theta & -c\phi c\psi & -c\phi s\psi s\theta - s\phi c\theta\end{bmatrix} \quad (24)$$

- The derivative related θ angle:

$$\frac{\partial\mathbf{A}_e}{\partial\theta} = \begin{bmatrix}-c\psi s\theta & 0 & -c\psi c\theta\\ c\phi s\psi s\theta + s\phi c\theta & 0 & c\phi s\psi c\theta - s\phi s\theta\\ -s\phi s\psi s\theta + c\phi c\theta & 0 & -s\phi s\psi c\theta - c\phi s\theta\end{bmatrix} \quad (25)$$

- The derivative related ψ angle:

$$\frac{\partial \mathbf{A}_e}{\partial \psi} = \begin{bmatrix} -s\psi c\theta & c\psi & s\psi s\theta \\ -c\phi c\psi c\theta & -c\phi s\psi & c\phi c\psi s\theta \\ s\phi c\psi c\theta & s\phi s\psi & -s\phi c\psi s\theta \end{bmatrix} \quad (26)$$

For the case where $\phi \to 0$; $\theta \to 0$; $\psi \to 0$, the matrices obtained above are:

$$\mathbf{A}_e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \frac{\partial \mathbf{A}_e}{\partial \phi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix};$$

$$\frac{\partial \mathbf{A}_e}{\partial \theta} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \frac{\partial \mathbf{A}_e}{\partial \psi} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (27)$$

In order to obtain the connection between the derivatives of Euler angles and components of rotation velocity in the body frame, starting with relation:

$$\boldsymbol{\omega}_{BR} = \dot{\boldsymbol{\psi}} + \dot{\boldsymbol{\theta}} + \dot{\phi}, \quad (28)$$

we can write:

$$\begin{bmatrix} p_B \\ q_B \\ r_B \end{bmatrix} = \mathbf{A}_{\phi,\psi,\theta} \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \mathbf{A}_{\phi,\psi} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + \mathbf{A}_{\phi} \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} = \mathbf{A}_{\phi,\psi} \begin{bmatrix} 0 \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}.$$

If we denote attitude vector:

$$\mathbf{a}_A = [\phi \quad \theta \quad \psi] \quad (29)$$

we obtain:

$$\boldsymbol{\omega}_{BR} = \mathbf{U}_A \dot{\mathbf{a}}_A . \quad (30)$$

where:

$$\mathbf{U}_A = \begin{bmatrix} 1 & s\psi & 0 \\ 0 & c\phi c\psi & s\phi \\ 0 & -s\phi c\psi & c\phi \end{bmatrix} \quad (31)$$

Denoting $\mathbf{W}_A$ the connection matrix:

$$\mathbf{W}_A = U_A^{-1} = \begin{bmatrix} 1 & -c\phi s\psi / c\psi & s\phi s\psi / c\psi \\ 0 & c\phi / c\psi & -s\phi / c\psi \\ 0 & s\phi & c\phi \end{bmatrix}. \quad (32)$$

we can finally write:

$$\dot{\mathbf{a}}_A = \mathbf{W}_A [p_B \quad q_B \quad r_B]^T , \quad (33)$$

Observation: *The connection matrix* $\mathbf{W}_A$ *can be singular for case when* $c\psi = 0$.

Considering (21) from (33) yield:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \mathbf{W}_A \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} - \mathbf{W}_A \mathbf{A}_e \begin{bmatrix} 0 \\ \omega_j \\ 0 \end{bmatrix} \quad (34)$$

Considering (29) finally we can write matrix relationship:

$$\dot{\mathbf{a}}_A = \mathbf{W}_A \boldsymbol{\omega}_{BI} - \mathbf{W}_A \mathbf{A}_e \boldsymbol{\omega}_{RI} \quad (35)$$

The derivatives of the connection matrix are:

$$\frac{\partial \mathbf{W}_A}{\partial \phi} = \begin{bmatrix} 0 & s\phi s\psi / c\psi & c\phi s\psi / c\psi \\ 0 & -s\phi / c\psi & -c\phi / c\psi \\ 0 & c\phi & -s\phi \end{bmatrix}$$

$$\frac{\partial \mathbf{W}_A}{\partial \theta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

$$\frac{\partial \mathbf{W}_A}{\partial \psi} = \begin{bmatrix} 0 & -c\phi / c^2\psi & s\phi / c^2\psi \\ 0 & c\phi s\psi / c^2\psi & -s\phi s\psi / c^2\psi \\ 0 & 0 & 0 \end{bmatrix} \quad (36)$$

For case $\phi \to 0$; $\theta \to 0$; $\psi \to 0$, matrices obtained above are:

$$\mathbf{W}_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \frac{\partial \mathbf{W}_A}{\partial \phi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix};$$

$$\frac{\partial \mathbf{W}_A}{\partial \theta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \frac{\partial \mathbf{W}_A}{\partial \psi} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (37)$$

Besides the kinematical relations with the attitude angles described above in the model we use the rotation angles that we will introduce next.

In paper [10] a group of three angles, called the rotation angles, were first introduced. The sizes were used to describe the aircraft movement.

Angles of rotation have the advantage that they can be measured easily on board of the space vehicle. Furthermore they retain the advantages of quaternion, removing singularity from kinematical equations written with the attitude angles (32). Also, allow the polynomial expression of the kinematical equations, an important advantage in building high-speed algorithms and easily implemented on hardware support. Angles of rotation retain the advantage of angles Euler type, that of being quantities directly measurable with a concrete physical meaning.

It is well known that a sequence of rotations of a rigid body with a fixed point can be replaced by a single rotation σ around an axis through the fixed point.

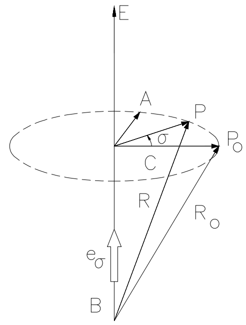

Fig. 3 Single rotation from fixed frame to mobile frame

In order to build kinematical equations we will use two frames:

$OX_0Y_0Z_0$ - The fixed frame with unitary vectors: $\mathbf{I},\mathbf{J},\mathbf{K}$;

$Oxyz$ – The mobile frame, linked by body with unitary vectors $\mathbf{i},\mathbf{j},\mathbf{k}$;

We suppose that the body has angular velocity $\boldsymbol{\omega}_{BR}$ with the components (p_B, q_B, r_B) in mobile frame *Oxyz:*

$$\boldsymbol{\omega}_{BR} = \mathbf{i}p_B + \mathbf{j}q_B + \mathbf{k}r_B . \qquad (38)$$

Axis E is the axis around which a single rotation σ is necessary to overlap frame $OX_0Y_0Z_0$ over frame *Oxyz* (fig. 3). The unitary vector for axis *E* is $\mathbf{e}_\sigma$:

$$\mathbf{e}_\sigma = \mathbf{I}l + \mathbf{J}m + \mathbf{K}n , \qquad (39)$$

Considering the notations from figure 3, we can write:

$$\mathbf{A} = \mathbf{e}_\sigma \times \mathbf{R}_0; \quad \mathbf{B} = \mathbf{e}_\sigma \cdot (\mathbf{e}_\sigma \cdot \mathbf{R}_0); \quad \mathbf{C} = \mathbf{R}_0 - \mathbf{B} . \qquad (40)$$

In this case, the relation between the position vectors of the point *P* and the point P_0 became successively:

$$\mathbf{R} = \mathbf{B} + \mathbf{A}\sin\sigma + \mathbf{C}\cos\sigma;$$

$$\mathbf{R} = \mathbf{e}_\sigma \cdot (\mathbf{e}_\sigma \cdot \mathbf{R}_0) + (\mathbf{e}_\sigma \times \mathbf{R}_0)\sin\sigma + [\mathbf{R}_0 - \mathbf{e}_\sigma \cdot (\mathbf{e}_\sigma \cdot \mathbf{R}_0)]\cos\sigma;$$

$$\mathbf{R} = \mathbf{R}_0 \cos\sigma + \mathbf{e}_\sigma \cdot (\mathbf{e}_\sigma \cdot \mathbf{R}_0)(1-\cos\sigma) + (\mathbf{e}_\sigma \times \mathbf{R}_0)\sin\sigma. \qquad (41)$$

If the point P_0 is located initially on the axis X_0 , the point *P* will be finally on-axis *x*.

Because the vectors $\mathbf{R}$ and $\mathbf{R}_0$ are equal in module, we can substitute in relation (41):

$$\mathbf{R} \to \mathbf{i}; \quad \mathbf{R}_0 \to \mathbf{I}.$$

Similarly, if the point P is located on the axis *y* or *z*, we can substitute:

$$\mathbf{R} \to \mathbf{j}; \quad \mathbf{R}_0 \to \mathbf{J}; \quad \mathbf{R} \to \mathbf{k}; \quad \mathbf{R}_0 \to \mathbf{K}.$$

Finally we obtain the system:

$$\mathbf{i} = \mathbf{I}\cos\sigma + (\mathbf{I}l + \mathbf{J}m + \mathbf{K}n)l(1-\cos\sigma) + (\mathbf{I}l + \mathbf{J}m + \mathbf{K}n)\times\mathbf{I}\sin\sigma;$$
$$\mathbf{j} = \mathbf{J}\cos\sigma + (\mathbf{I}l + \mathbf{J}m + \mathbf{K}n)m(1-\cos\sigma) + (\mathbf{I}l + \mathbf{J}m + \mathbf{K}n)\times\mathbf{J}\sin\sigma;$$
$$\mathbf{k} = \mathbf{K}\cos\sigma + (\mathbf{I}l + \mathbf{J}m + \mathbf{K}n)n(1-\cos\sigma) + (\mathbf{I}l + \mathbf{J}m + \mathbf{K}n)\times\mathbf{K}\sin\sigma. \qquad (42)$$

If we note $c = \cos\sigma; \quad s = \sin\sigma$, we obtain the relation:

$$[\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}]^T = \mathbf{A}_e [\mathbf{I} \quad \mathbf{J} \quad \mathbf{K}]^T ,$$

where, $\mathbf{A}_e$ is the direct rotation matrix:

$$\mathbf{A}_e = \begin{bmatrix} c + l^2(1-c) & lm(1-c) + ns & ln(1-c) - ms \\ lm(1-c) - ns & c + m^2(1-c) & mn(1-c) + ls \\ ln(1-c) + ms & mn(1-c) - ls & c + n^2(1-c) \end{bmatrix} \qquad (43)$$

which coincides with that defined by the relation (23) using attitude angles.

Thus, the overall rotation angle can be expressed by the superposition of three simultaneous rotations along the mobile frame axes:

$$\xi = \sigma l; \quad \eta = \sigma m; \quad \zeta = \sigma n \qquad (44)$$

The sizes are called the rotation angles:

ξ - Rotation angle around x axis;

η - Rotation angle around y axis;

ζ - Rotation angle around z axis.

The angles verify the relation:

$$\sigma^2 = \xi^2 + \eta^2 + \zeta^2 . \qquad (45)$$

Using rotation angles, from (43) the rotation matrix becomes:

$$\mathbf{A}_e = \begin{bmatrix} a\xi^2 + c & a\eta\xi + b\zeta & a\zeta\xi - b\eta \\ a\xi\eta - b\zeta & a\eta^2 + c & a\zeta\eta + b\xi \\ a\xi\zeta + b\eta & a\eta\zeta - b\xi & a\zeta^2 + c \end{bmatrix} \qquad (46)$$

where:

$$a = \frac{1-c}{\sigma^2}; \quad b = \frac{s}{\sigma}; \quad c = \cos\sigma; \; s = \sin\sigma \qquad (47)$$

From (45) we obtain derivatives:

$$\frac{\partial\sigma}{\partial\xi} = \frac{\xi}{\sigma}; \quad \frac{\partial\sigma}{\partial\eta} = \frac{\eta}{\sigma}; \quad \frac{\partial\sigma}{\partial\zeta} = \frac{\zeta}{\sigma}, \qquad (48)$$

with which we obtain derivatives matrix $\mathbf{A}_e$:

$$\frac{\partial\mathbf{A}_e}{\partial\xi} = \begin{bmatrix} 2a\xi & a\eta & a\zeta \\ a\eta & 0 & b \\ a\zeta & -b & 0 \end{bmatrix} + \frac{\partial\mathbf{A}_e}{\partial\sigma}\frac{\xi}{\sigma};$$

$$\frac{\partial\mathbf{A}_e}{\partial\eta} = \begin{bmatrix} 0 & a\xi & -b \\ a\xi & 2a\eta & a\zeta \\ b & a\zeta & 0 \end{bmatrix} + \frac{\partial\mathbf{A}_e}{\partial\sigma}\frac{\eta}{\sigma}$$

$$\frac{\partial\mathbf{A}_e}{\partial\zeta} = \begin{bmatrix} 0 & b & a\xi \\ -b & 0 & a\eta \\ a\xi & -a\eta & 2a\zeta \end{bmatrix} + \frac{\partial\mathbf{A}_e}{\partial\sigma}\frac{\zeta}{\sigma} \qquad (49)$$

where:

$$\frac{\partial \mathbf{A}_e}{\partial \sigma}=\begin{bmatrix} a'\xi^2-s & a'\eta\xi+b'\zeta & a'\zeta\xi-b'\eta \\ a'\xi\eta-b'\zeta & a'\eta^2-s & a'\zeta\eta+b'\xi \\ a'\xi\zeta+b'\eta & a'\eta\zeta-b'\xi & a'\zeta^2-s \end{bmatrix} \quad (50)$$

where we noted

$$a'=\frac{da}{d\sigma}=\frac{\sigma s+2c-2}{\sigma^3};\ b'=\frac{db}{d\sigma}=\frac{\sigma c-s}{\sigma^2}; \quad (51)$$

An interesting case is $\sigma \to 0$.

For this situation we have:

$$\lim_{\sigma\to 0} a=\frac{1}{2};\ \lim_{\sigma\to 0} b=1;\ \lim_{\sigma\to 0} a'=0\ \ \lim_{\sigma\to 0} b'=0; \quad (52)$$

and matrix derivatives are:

$$\mathbf{A}_e=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix};\ \frac{\partial \mathbf{A}_e}{\partial \sigma}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix};$$

$$\frac{\partial \mathbf{A}_e}{\partial \xi}=\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix};\frac{\partial \mathbf{A}_e}{\partial \eta}=\begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix};$$

$$\frac{\partial \mathbf{A}_e}{\partial \zeta}=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix} \quad (53)$$

Because the rotation matrix (23) and (46) is the same regardless of the variables used, we obtain the following relationships between different variables (Euler angles, rotation angles)

The attitude angles from the rotation angles are given by:

$$\tan\phi=-\frac{a_{3,2}}{a_{2,2}}=\frac{b\xi-a\zeta\eta}{a\eta^2+c};\tan\theta=\frac{a_{1,3}}{a_{1,1}}=\frac{b\eta-a\zeta\xi}{a\xi^2+c}$$
$$\sin\psi=-a_{1,2}=b\zeta+a\eta\xi; \quad (54)$$

Also, we can obtain the rotation angles from attitude angles using the relations:

$$\xi=\frac{a_{2,3}-a_{3,2}}{2b};\ \eta=\frac{a_{3,1}-a_{1,3}}{2b}\ \ \zeta=\frac{a_{1,2}-a_{2,1}}{2b} \quad (55)$$

where:

$$b=\frac{\sin\sigma}{\sigma};\ \sigma=\arccos c\ \ c=\left(a_{1,1}+a_{2,2}+a_{3,3}-1\right)/2. \quad (56)$$

Next, we will try to obtain the connection between the derivatives of rotation angles and components of rotation velocity in the body frame.

Thus, as rotation around axis E is an equivalent transformation in terms of the two systems, it follows that the vector $\mathbf{e}_\sigma$ projections are identical:

$$\mathbf{e}_\sigma=\mathbf{I}l+\mathbf{J}m+\mathbf{K}n=\mathbf{i}l+\mathbf{j}m+\mathbf{k}n\ . \quad (57)$$

If this relationship is derived with respect to time we obtain:

$$\mathbf{I}\dot{l}+\mathbf{J}\dot{m}+\mathbf{K}\dot{n}=\mathbf{i}\dot{l}+\mathbf{j}\dot{m}+\mathbf{k}\dot{n}+\boldsymbol{\omega}_{BR}\times\mathbf{e}_\sigma, \quad (58)$$

where:

$$\boldsymbol{\omega}_{BR}\times\mathbf{e}_\sigma=\mathbf{i}(q_Bn-r_Bm)+\mathbf{j}(r_Bl-p_Bn)+\mathbf{k}(p_Bm-q_Bl), \quad (59)$$

thus:

$$\mathbf{I}\dot{l}+\mathbf{J}\dot{m}+\mathbf{K}\dot{n}=\mathbf{i}(\dot{l}+q_Bn-r_Bm)+$$
$$+\mathbf{j}(\dot{m}+r_Bl-p_Bn)+\mathbf{k}(\dot{n}+p_Bm-q_Bl). \quad (60)$$

If we multiply successively by $\mathbf{i},\mathbf{j},\mathbf{k}$ results:

$$\begin{bmatrix}\mathbf{I}\cdot\mathbf{i}&\mathbf{J}\cdot\mathbf{i}&\mathbf{K}\cdot\mathbf{i}\\\mathbf{I}\cdot\mathbf{j}&\mathbf{J}\cdot\mathbf{j}&\mathbf{K}\cdot\mathbf{j}\\\mathbf{I}\cdot\mathbf{k}&\mathbf{J}\cdot\mathbf{k}&\mathbf{K}\cdot\mathbf{k}\end{bmatrix}\begin{bmatrix}\dot{l}\\\dot{m}\\\dot{n}\end{bmatrix}=\begin{bmatrix}\dot{l}\\\dot{m}\\\dot{n}\end{bmatrix}+\begin{bmatrix}0&n&-m\\-n&0&l\\m&-l&0\end{bmatrix}\begin{bmatrix}p_B\\q_B\\r_B\end{bmatrix},$$

or otherwise:

$$\left[\mathbf{I}-\mathbf{A}_e\right]\begin{bmatrix}\dot{l}\\\dot{m}\\\dot{n}\end{bmatrix}=\begin{bmatrix}0&-n&m\\n&0&-l\\-m&l&0\end{bmatrix}\begin{bmatrix}p_B\\q_B\\r_B\end{bmatrix}. \quad (61)$$

Introducing the matrix $\mathbf{A}_e$ given by (46), the left member of the relationship becomes:

$$\begin{bmatrix}1-c&-ns&ms\\ns&1-c&-ls\\-ms&ls&1-c\end{bmatrix}\begin{bmatrix}\dot{l}\\\dot{m}\\\dot{n}\end{bmatrix}-(1-c)\begin{bmatrix}l^2&lm&nl\\lm&m^2&mn\\nl&mn&n^2\end{bmatrix}\begin{bmatrix}\dot{l}\\\dot{m}\\\dot{n}\end{bmatrix}\equiv$$
$$\equiv\frac{2t}{1+t^2}\begin{bmatrix}t&-n&m\\n&t&-l\\-m&l&t\end{bmatrix}\begin{bmatrix}\dot{l}\\\dot{m}\\\dot{n}\end{bmatrix}-(1-c)(l\dot{l}+m\dot{m}+n\dot{n})\begin{bmatrix}l\\m\\n\end{bmatrix}, \quad (62)$$

where we noted $t=\mathrm{tg}(\sigma/2)$

Since the projections of unitary vector $\mathbf{e}_\sigma$ satisfying the relationship:

$$l^2+m^2+n^2=1, \quad (51)$$

results from differentiation:

$$l\dot{l}+m\dot{m}+n\dot{n}=0, \quad (63)$$

making the last term of the previous development to be null.

On the other hand reverse matrix of the first term of relation (62) is

$$\begin{bmatrix}t&-n&m\\n&t&-l\\-m&l&t\end{bmatrix}^{-1}=\frac{1}{t(1+t^2)}\begin{bmatrix}l^2+t^2&lm+nt&nl-mt\\lm-nt&m^2+t^2&mn+lt\\nl+mt&mn-lt&n^2+t^2\end{bmatrix}. \quad (64)$$

Multiplying by inverse matrix thus defined, relation (62) becomes:

$$2t\begin{bmatrix}\dot{l}\\ \dot{m}\\ \dot{n}\end{bmatrix}=\begin{bmatrix}1-l^2 & -lm-nt & -nl+mt\\ -lm+nt & 1-m^2 & -mn-lt\\ -nl-mt & -mn+lt & 1-n^2\end{bmatrix}\begin{bmatrix}p_B\\ q_B\\ r_B\end{bmatrix}, \quad (65)$$

which leads to algebraic relations:

$$\begin{aligned}2\dot{l} &= r_B m - q_B n + (p_B - l\dot{\sigma})/t;\\ 2\dot{m} &= p_B n - r_B l + (q_B - m\dot{\sigma})/t;\\ 2\dot{n} &= q_B l - p_B m + (r_B - n\dot{\sigma})/t,\end{aligned} \quad (66)$$

where:

$$\dot{\sigma} = \mathbf{\Omega}\cdot\mathbf{e}_\sigma = p_B l + q_B m + r_B n \ . \quad (67)$$

By derivation of the definition relations (44) we obtain:

$$\begin{aligned}\dot{\xi} &= (1-h)l\dot{\sigma} + h(p_B - ntq_B + mtr_B);\\ \dot{\eta} &= (1-h)m\dot{\sigma} + h(ntp_B + q_B - ltr_B);\\ \dot{\varsigma} &= (1-h)n\dot{\sigma} + h(-mtp_B + ltq_B + r_B),\end{aligned} \quad (68)$$

where derivatives of the angles of rotation can be put in the form:

$$\begin{bmatrix}\dot{\xi} & \dot{\eta} & \dot{\zeta}\end{bmatrix}^T = \mathbf{W}_R\begin{bmatrix}p_B & q_B & r_B\end{bmatrix}^T \quad (69)$$

in which, with local notations:

$$h=\frac{\sigma}{2t};\quad f=\frac{1-h}{\sigma^2}, \quad (70)$$

connection matrix $\mathbf{W}_R$ is given by:

$$\mathbf{W}_R = f\begin{bmatrix}\xi^2 & \eta\xi & \zeta\xi\\ \xi\eta & \eta^2 & \zeta\eta\\ \xi\zeta & \eta\zeta & \zeta^2\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0 & -\zeta & \eta\\ \zeta & 0 & -\xi\\ -\eta & \xi & 0\end{bmatrix}+h\mathbf{I}, \quad (71)$$

or, in compact form:

$$\mathbf{W}_R=\begin{bmatrix}f\xi^2+h & f\eta\xi-\zeta/2 & f\zeta\xi+\eta/2\\ f\xi\eta+\zeta/2 & f\eta^2+h & f\zeta\eta-\xi/2\\ f\xi\zeta-\eta/2 & f\eta\zeta+\xi/2 & f\zeta^2+h\end{bmatrix} \quad (72)$$

If we denote:

$$\mathbf{a}_R = \begin{bmatrix}\xi & \eta & \zeta\end{bmatrix}^T$$

we can write relation (69) in following form:

$$\dot{\mathbf{a}}_R = \mathbf{W}_R\boldsymbol{\omega}_{BR} \quad (73)$$

Relation (73) represents kinematical equations written using rotation angles, being equivalent with relation (33) which is written using attitude angles.

Using notations:

$$h'=\frac{dh}{d\sigma}=\frac{1}{2}\frac{s-\sigma}{1-c};\quad f'=\frac{df}{d\sigma}=\frac{\sigma^2+\sigma s-4(1-c)}{2\sigma^3(1-c)}, \quad (74)$$

we can determinate the derivatives of the matrix $\mathbf{W}_R$:

$$\frac{\partial\mathbf{W}_R}{\partial\xi}=\begin{bmatrix}2f\xi & f\eta & f\zeta\\ f\eta & 0 & -1/2\\ f\zeta & 1/2 & 0\end{bmatrix}+\frac{\partial\mathbf{W}_R}{\partial\sigma}\frac{\xi}{\sigma};$$

$$\frac{\partial\mathbf{W}_R}{\partial\eta}=\begin{bmatrix}0 & f\xi & 1/2\\ f\xi & 2f\eta & f\zeta\\ -1/2 & f\zeta & 0\end{bmatrix}+\frac{\partial\mathbf{W}_R}{\partial\sigma}\frac{\eta}{\sigma};$$

$$\frac{\partial\mathbf{W}_R}{\partial\zeta}=\begin{bmatrix}0 & -1/2 & f\zeta\\ 1/2 & 0 & f\eta\\ f\xi & f\eta & 2f\zeta\end{bmatrix}+\frac{\partial\mathbf{W}_R}{\partial\sigma}\frac{\zeta}{\sigma}, \quad (75)$$

where:

$$\frac{\partial\mathbf{W}_R}{\partial\sigma}=\begin{bmatrix}f'\xi^2+h' & f'\eta\xi & f'\zeta\xi\\ f'\xi\eta & f'\eta^2+h' & f'\zeta\eta\\ f'\xi\zeta & f'\eta\zeta & f'\zeta^2+h'\end{bmatrix}. \quad (76)$$

An interesting case is when $\sigma\to 0$.

For this situation we have:

$$\lim_{\sigma\to 0}h=1 \ \lim_{\sigma\to 0}f=\frac{1}{12};\ \lim_{\sigma\to 0}h'=0 \ \lim_{\sigma\to 0}f'=0 \quad (77)$$

and the derivatives are:

$$\mathbf{W}_R=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix};\ \frac{\partial\mathbf{W}_R}{\partial\sigma}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix};$$

$$\frac{\partial\mathbf{W}_R}{\partial\xi}=\begin{bmatrix}0&0&0\\0&0&-1/2\\0&1/2&0\end{bmatrix};\ \frac{\partial\mathbf{W}_R}{\partial\eta}=\begin{bmatrix}0&0&1/2\\0&0&0\\-1/2&0&0\end{bmatrix};$$

$$\frac{\partial\mathbf{W}_R}{\partial\zeta}=\begin{bmatrix}0&-1/2&0\\1/2&0&0\\0&0&0\end{bmatrix}. \quad (78)$$

Beside equations (33) or (73) which describe vehicle orientation there are still three equations that starting from dynamic Newton equation, linear coordinates of the vehicles will produce:

$$\dot{x}_I = V_{Ix};\ \dot{y}_I = V_{Iy} \ \dot{z}_I = V_{Iz} \quad (79)$$

D. Dynamical equations

Developing vector equations presented in the paper [4], and considering that the satellite has no moving parts and weight is constant, we can write two matrix equations

- Force equations in the Earth frame

$$\dot{V}_{Ix} = -\frac{\mu}{r^3}x_I \text{ ; } \dot{V}_{Iy} = -\frac{\mu}{r^3}y_I \text{ ; } \dot{V}_{Iz} = -\frac{\mu}{r^3}z_I \text{ ;} \quad (80)$$

where:

$$r = \sqrt{x_I^2 + y_I^2 + z_I^2} \quad (81)$$

- Moment equations in the body frame, relations known as Euler dinamic equations:

$$\begin{bmatrix}\dot{\omega}_x\\ \dot{\omega}_y\\ \dot{\omega}_z\end{bmatrix} = \mathbf{J}^{-1}\begin{bmatrix}L_G\\ M_G\\ N_G\end{bmatrix} + \mathbf{J}^{-1}\begin{bmatrix}L_C\\ M_C\\ N_C\end{bmatrix} + \mathbf{J}^{-1}\begin{bmatrix}(B-C)\omega_y\omega_z + E\omega_x\omega_y\\ (C-A)\omega_z\omega_x + E(\omega_z^2 - \omega_x^2)\\ (A-B)\omega_x\omega_y - E\omega_y\omega_z\end{bmatrix} \quad (82)$$

where two inertial products are null:

$$D = F = 0 \quad (82)$$

The inverse matrix for the inertia moment is given by:

$$\mathbf{J}^{-1} = \frac{1}{AC - E^2}\begin{bmatrix}C & 0 & E\\ 0 & (AC - E^2)/B & 0\\ E & 0 & A\end{bmatrix} \quad (84)$$

a

nd the inertial moments are given by:

$$A = \int (y^2 + z^2)\,\mathrm{d}m; \; B = \int (z^2 + x^2)\,\mathrm{d}m; \; C = \int (x^2 + y^2)\,\mathrm{d}m$$

$$E = \int zx\,\mathrm{d}m; \quad (86)$$

The moment applied to vehicle has two terms:

- Gravitationally moment term:

$$\mathbf{M}_G = \begin{bmatrix}L_G & M_G & N_G\end{bmatrix}^T \quad (87)$$

-

Command term which is performed using micro-jet:

$$\mathbf{M}_C = \begin{bmatrix}L_C & M_C & N_C\end{bmatrix}^T \quad (88)$$

State vector for these equations is:

$$\boldsymbol{\omega}_{BI} = \begin{bmatrix}\omega_x & \omega_y & \omega_z\end{bmatrix}^T, \quad (89)$$

and means the rotation speed of the body frame related to the inertial frame, heaving components along body frame. These nonlinear differential equations have no closed analytical solution.

F. Gravitational Moment

Space vehicle has a asymmetric body, situation where there is a tendency to align its principal axes of inertia according to the direction of the gravitational field.

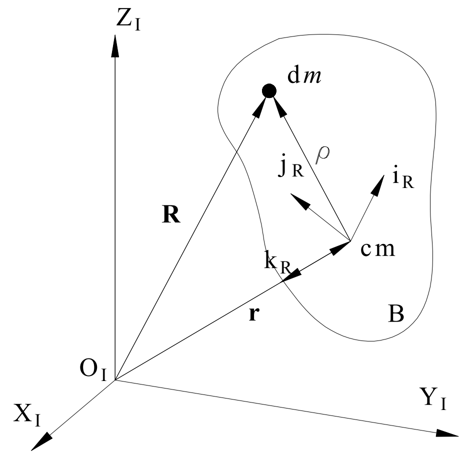

Fig. 4 gravitational moment of mass element

If we assume that we have a vehicle, whose center of mass (cm) is positioned at a distance $\mathbf{r}$ from Earth's center, and a mass element $\mathrm{d}m$ belonging to the vehicul positioned at a distance $\boldsymbol{\rho}$ from the vehicul center of mass and a distance $\mathbf{R}$ from the center of the Earth we can write the link between them:

$$\mathbf{R} = \mathbf{r} + \boldsymbol{\rho} \quad (90)$$

In the reference frame, the position vector for the center of mass has the form:

$$\mathbf{r} = -\mathbf{k}_R r \quad (91)$$

If we wish to express this vector in the body frame we have:

$$\begin{bmatrix}r_x & r_y & r_z\end{bmatrix}^T = \mathbf{A}_A\begin{bmatrix}0 & 0 & -r\end{bmatrix}^T, \quad (92)$$

from which:

$$r_x = -ra_{1,3}\text{ ; } r_y = -ra_{2,3}\text{ ; } r_z = -ra_{3,3} \quad (93)$$

The position vector for the mass element is given by:

$$\boldsymbol{\rho} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad (94)$$

Mass element involve following elementary gravitational moment:

$$\mathrm{d}\mathbf{M}_G = \boldsymbol{\rho}\times\mathrm{d}\mathbf{G} = -\frac{\mu\,\mathrm{d}m}{R^3}\boldsymbol{\rho}\times\mathbf{R} = -\frac{\mu\,\mathrm{d}m}{R^3}\boldsymbol{\rho}\times\mathbf{r} \quad (95)$$

For R^{-3}, taking into account that: $\rho < R$ and $\rho < r$, we can successively write:

$$R^2 = r^2 + \rho^2 + 2\boldsymbol{\rho}\mathbf{r} \cong r^2\left(1 + 2\frac{\boldsymbol{\rho}\mathbf{r}}{r^2}\right) \quad (96)$$

from where:

$$\frac{1}{R^3} \cong \frac{1}{r^3}\left(1 - 3\frac{\boldsymbol{\rho}\mathbf{r}}{r^2}\right) \quad (97)$$

In this case, gravitational moment becomes:

$$\mathbf{M}_G = -\mu\int_{\mathbf{m}}\frac{\mathrm{d}m}{R^3}\boldsymbol{\rho}\times\mathbf{r} \cong -\frac{\mu}{r^3}\int_{\mathbf{m}}\left(1 - 3\frac{\boldsymbol{\rho}\mathbf{r}}{r^2}\right)\boldsymbol{\rho}\times\mathbf{r}\,dm \quad (98)$$

$$\mathbf{M}_G \cong \frac{3\mu}{r^5}\int_{\mathbf{m}}(\boldsymbol{\rho}\mathbf{r})(\boldsymbol{\rho}\times\mathbf{r})dm \quad (99)$$

Scalar product becomes:

$$\boldsymbol{\rho}\mathbf{r} = -r(a_{13}x + a_{23}y + a_{33}z) \quad (100)$$

and vectorial product is:

$$\boldsymbol{\rho}\times\mathbf{r} = -r(a_{3,3}y - a_{2,3}z)\mathbf{i} - r(a_{1,3}z - a_{3,3}x)\mathbf{j} - r(a_{2,3}x - a_{1,3}y)\mathbf{k} \quad (101)$$

and integral expression, neglected inertial products becomes:

$$\mathbf{M}_G \cong \frac{3\mu}{r^3}[(B-C)a_{2,3}a_{3,3}\mathbf{i} + (C-A)a_{1,3}a_{3,3}\mathbf{j} + (A-B)a_{1,3}a_{2,3}\mathbf{k}] \quad (102)$$

In order to evaluate the coefficient size we can express the coefficient $3\mu/r^3$ using angular velocity ω_j:

$$3\frac{\mu}{r^3} = 3\frac{h^2 r}{(1-e^2)ar^4} = 3\frac{\omega_j^2 r}{(1-e^2)a} \quad (103)$$

In this case, the gravity gradient moment components become:

$$L_G = 3\frac{\omega_j^2 r}{(1-e^2)a}(B-C)a_{2,3}a_{3,3};$$
$$M_G = 3\frac{\omega_j^2 r}{(1-e^2)a}(C-A)a_{1,3}a_{3,3}$$
$$N_G = 3\frac{\omega_j^2 r}{(1-e^2)a}(A-B)a_{1,3}a_{2,3} \quad (104)$$

If we consider attitude angles, the relations are:

$$L_G = 3\frac{\omega_j^2 r}{(1-e^2)a}(B-C)\times \times(c\phi s\psi s\theta + s\phi c\theta)(-s\phi s\psi s\theta + c\phi c\theta);$$
$$M_G = 3\frac{\omega_j^2 r}{(1-e^2)a}(C-A)\times \times(-c\psi s\theta)(-s\phi s\psi s\theta + c\phi c\theta)$$
$$N_G = 3\frac{\omega_j^2 r}{(1-e^2)a}(A-B)\times (-c\psi s\theta)\times(c\phi s\psi s\theta + s\phi c\theta) \quad (105)$$

If we consider rotation angles we will obtain:

$$L_G = 3\frac{\omega_j^2 r}{(1-e^2)a}(B-C)(a\zeta\eta + b\xi)(a\zeta^2 + c)$$
$$M_G = 3\frac{\omega_j^2 r}{(1-e^2)a}(C-A)(a\zeta\xi - b\eta)(a\zeta^2 + c)$$
$$N_G = 3\frac{\omega_j^2 r}{(1-e^2)a}(A-B)(a\zeta\xi - b\eta)(a\zeta\eta + b\xi) \quad (106)$$

III. Auxiliary equations

For guidance command we need integrals term defined hereby:

$$\begin{bmatrix}\dot{I}_\phi & \dot{I}_\theta & \dot{I}_\psi\end{bmatrix} = \begin{bmatrix}\phi & \theta & \psi\end{bmatrix}. \quad (107)$$

or, if we use rotation angles:

$$\begin{bmatrix}\Delta\dot{I}_\xi & \Delta\dot{I}_\eta & \Delta\dot{I}_\zeta\end{bmatrix}^T = \begin{bmatrix}\Delta\xi & \Delta\eta & \Delta\zeta\end{bmatrix}^T \quad (108)$$

Because the satellite's orientation control by means of engines is due by their symmetrical arrangement about the rotation axis, applying torque in either direction can not be done only by switching between two motors. For it is necessary that the chain of command to contain a switching element to achieve a discrete output, constant amplitude, modulated in duration. As shown in [7], the control system can be described by a Schmidt trigger type element, whose functional diagram is given in Figure 5. It is noted that this element is composed of a nonlinear block, relay with hysteresis and insensitivity zone and a linear integrator block that allows additional tuning of the system. To control the output, this is turned to the entry, forming a feedback loop.

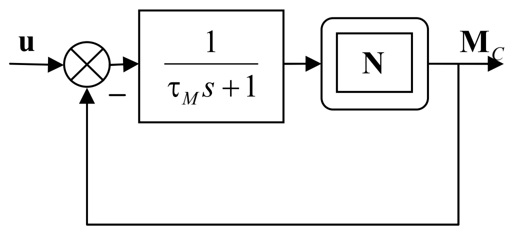

Fig. 5 command type trigger Schmidt with nonlinear element

Nonlinear element is relay type, with insensitivity zone and hysteresis, as we can see in figure 6

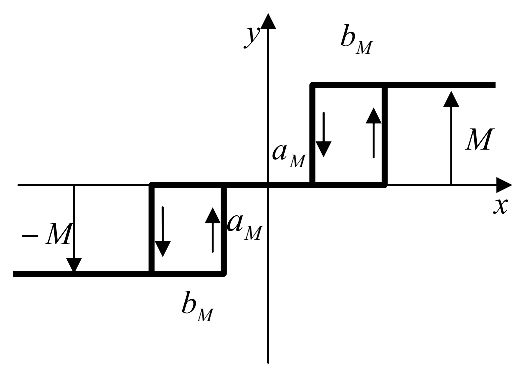

Fig. 6 nonlinear element operating schedule (N)

As we can see in figure 6 the size of insensitivity zone is $2a$, the size of hysteresis zone is b, and saturation command is $\pm M$ where τ_M are time constants and k_M^u gain constants.

IV. Balance mouvement

The study of flight stability will be made accordingly to Liapunov theory, considering the system of movement equations perturbed around the balanced movement. This involves a disturbance shortly applied on the balance movement, which will produce deviation of the state variables. Developing in series the perturbed movement equations in relation to status variables and taking into account the first order terms of the detention, we will get linear equations which can be used to analyze the stability in the first approximation, as we proceed in most dynamic non linear problems. To determine basic movement parameters in equations we consider the vehicle stabilized with the body frame overlap reference frame. That means the attitude angles or the rotation angles are nulls:

$$\mathbf{a}_A = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T ;\ \mathbf{a}_R = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \quad (109)$$

and also the angular velocity or body frame related reference frame are nulls.

$$\boldsymbol{\omega}_{BR} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \quad (110)$$

In this case the link between angular velocity, for balance movement becomes:

$$\boldsymbol{\omega}_{BI} = \boldsymbol{\omega}_{RIB}, \quad (111)$$

moreover, because the attitude or rotation angles are nulls, and rotation matrix $\mathbf{A}_e$ is a unitary matrix, the previously relation becomes:

$$\boldsymbol{\omega}_{BI} = \boldsymbol{\omega}_{RI}, \quad (113)$$

or, in scalar form:

$$\omega_x = 0;\ \omega_y = \omega_j;\ \omega_z = 0 \quad (113)$$

In order to have a stationary movement, we admit that the orbit is circular. This hypothesis leads to a constant orbit range, $r = a$, and allows us to have a constant value for orbital angular velocity:

$$\omega_j = -\frac{v}{r} \quad (114)$$

V. Linear form of the general equations

Considering base general equation due by Kepler model we can obtain linear form.

From dynamic Euler equation we obtain following linear form:

$$\Delta\dot{\boldsymbol{\omega}}_{BI} = \mathbf{M}_\omega \Delta\boldsymbol{\omega}_{BI} + \mathbf{M}_R \Delta\mathbf{a}_R + \mathbf{J}^{-1}\Delta\mathbf{M} \quad (115)$$

where:

$$\mathbf{M}_\omega = \mathbf{J}^{-1}\begin{bmatrix} B-C & 0 & 0 \\ 0 & C-A & 0 \\ 0 & 0 & A-B \end{bmatrix}\begin{bmatrix} 0 & \omega_z & \omega_y \\ \omega_z & 0 & \omega_x \\ \omega_y & \omega_x & 0 \end{bmatrix} + \mathbf{J}^{-1}E\begin{bmatrix} \omega_y & \omega_x & 0 \\ -2\omega_x & 0 & 2\omega_z \\ 0 & -\omega_z & -\omega_y \end{bmatrix} \quad (116)$$

Taking into account balance movement established above, the matrix becomes:

$$\mathbf{M}_\omega = \omega_j \begin{bmatrix} \frac{-E(B-A-C)}{AC-E^2} & 0 & \frac{BC-C^2-E^2}{AC-E^2} \\ 0 & 0 & 0 \\ \frac{E^2+A^2-AB}{AC-E^2} & 0 & \frac{E(B-A-C)}{AC-E^2} \end{bmatrix} \quad (117)$$

$$\Delta\mathbf{M}_C = \begin{bmatrix} \Delta L_C & \Delta M_C & \Delta N_C \end{bmatrix}^T \quad (118)$$

Starting from gravity gradient moment components, for small attitude angles we obtain the following relations:

$$\Delta L_g = 3\omega_j^2 (B-C)\Delta\phi$$
$$\Delta M_g = 3\omega_j^2 (A-C)\Delta\theta$$
$$\Delta N_g = 0 \quad (119)$$

or, if we use rotation angles:

$$\Delta L_g = 3\omega_j^2 (B-C)\Delta\xi$$
$$\Delta M_g = 3\omega_j^2 (A-C)\Delta\eta$$
$$\Delta N_g = 0 \quad (120)$$

Matrix form becomes in both cases:

$$\mathbf{M}_R = 3\omega_j^2 \mathbf{J}^{-1}\begin{bmatrix} B-C & 0 & 0 \\ 0 & A-C & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (121)$$

or:

$$\mathbf{M}_R = 3\omega_j^2 \begin{bmatrix} \frac{C(B-C)}{AC-E^2} & 0 & 0 \\ 0 & \frac{A-C}{B} & 0 \\ \frac{E(B-C)}{AC-E^2} & 0 & 0 \end{bmatrix} \quad (122)$$

From kinematical equation (33), if we use altitude angles we obtain:

$$\Delta\dot{\mathbf{a}}_A = \mathbf{W}_A \Delta\boldsymbol{\omega}_{BR} + \mathbf{W}_{\omega A}\Delta\mathbf{a}_A \quad (123)$$

where:

$$\mathbf{W}_{\omega A} = \begin{bmatrix} \frac{\partial \mathbf{W}_A}{\partial \phi} & \vdots & \frac{\partial \mathbf{W}_A}{\partial \theta} & \vdots & \frac{\partial \mathbf{W}_A}{\partial \psi} \end{bmatrix}\begin{bmatrix} \boldsymbol{\omega}_{BR} & 0 & 0 \\ 0 & \boldsymbol{\omega}_{BR} & 0 \\ 0 & 0 & \boldsymbol{\omega}_{BR} \end{bmatrix} \quad (124)$$

For the base movement, because we consider $\boldsymbol{\omega}_{BR} = 0$, that lends to a nulls matrix:

$$\mathbf{W}_{\omega A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (125)$$

Taking into account that $\boldsymbol{\omega}_{RI}$ is base movement, it leads to

$$\Delta\omega_{RI} = 0. \quad (126)$$

In this case, the relation in perturbations between angular velocity becomes:

$$\Delta\boldsymbol{\omega}_{BR} = \Delta\boldsymbol{\omega}_{BI} - \mathbf{A}_{e\omega A}\Delta\mathbf{a}_A \quad (127)$$

where:

$$\mathbf{A}_{e\omega A} = \left[\frac{\partial \mathbf{A}_e}{\partial \phi} \;\vdots\; \frac{\partial \mathbf{A}_e}{\partial \theta} \;\vdots\; \frac{\partial \mathbf{A}_e}{\partial \psi} \right] \begin{bmatrix} \boldsymbol{\omega}_{RI} & 0 & 0 \\ 0 & \boldsymbol{\omega}_{RI} & 0 \\ 0 & 0 & \boldsymbol{\omega}_{RI} \end{bmatrix} \quad (128)$$

For the base movement we have:

$$\mathbf{A}_{e\omega A} = \begin{bmatrix} 0 & 0 & \omega_j \\ 0 & 0 & 0 \\ -\omega_j & 0 & 0 \end{bmatrix} \quad (129)$$

From relation (123) it results:

$$\Delta \dot{\mathbf{a}}_A = \mathbf{W}_A \Delta \boldsymbol{\omega}_{BI} + \mathbf{A}_A \Delta \mathbf{a}_A \quad (130)$$

where:

$$\mathbf{A}_A = \mathbf{W}_{\omega A} - \mathbf{W}_A \mathbf{A}_{e\omega A} \quad (131)$$

For the base movement we have:

$$\mathbf{A}_A = -\mathbf{A}_{e\omega A} = \begin{bmatrix} 0 & 0 & -\omega_j \\ 0 & 0 & 0 \\ \omega_j & 0 & 0 \end{bmatrix} \quad (132)$$

Similarly, if we use rotation angles, the kinematical equation (73) in linear form becomes:

$$\Delta \dot{\mathbf{a}}_R = \mathbf{W}_R \Delta \boldsymbol{\omega}_{BI} + \mathbf{A}_R \Delta \mathbf{a}_R \quad (133)$$

where:

$$\mathbf{A}_R = \mathbf{W}_{\omega R} - \mathbf{W}_R \mathbf{A}_{e\omega R} \quad (134)$$

whereabouts:

$$\mathbf{W}_{\omega R} = \left[\frac{\partial \mathbf{W}_A}{\partial \xi} \;\vdots\; \frac{\partial \mathbf{W}_A}{\partial \eta} \;\vdots\; \frac{\partial \mathbf{W}_A}{\partial \zeta} \right] \begin{bmatrix} \boldsymbol{\omega}_{BR} & 0 & 0 \\ 0 & \boldsymbol{\omega}_{BR} & 0 \\ 0 & 0 & \boldsymbol{\omega}_{BR} \end{bmatrix}$$

$$\mathbf{A}_{e\omega R} = \left[\frac{\partial \mathbf{A}_e}{\partial \xi} \;\vdots\; \frac{\partial \mathbf{A}_e}{\partial \eta} \;\vdots\; \frac{\partial \mathbf{A}_e}{\partial \zeta} \right] \begin{bmatrix} \boldsymbol{\omega}_{RI} & 0 & 0 \\ 0 & \boldsymbol{\omega}_{RI} & 0 \\ 0 & 0 & \boldsymbol{\omega}_{RI} \end{bmatrix} \quad (135)$$

Finally, using relations (115) , (133), rotation angles cases we can outline the stability and the command matrices:

Table 1 stability matrix

	$\boldsymbol{\omega}_{BI}$	$\mathbf{a}_R$
$\boldsymbol{\omega}_{BI}$	$\mathbf{M}_\omega$	$\mathbf{M}_R$
$\mathbf{a}_R$	$\mathbf{W}_R$	$\mathbf{A}_R$

Table 2 command matrix

	$\boldsymbol{\omega}_{BI}$
$\boldsymbol{\omega}_{BI}$	$\mathbf{J}^{-1}$
$\mathbf{a}_R$	

In this case the system can be put in standard form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (136)$$

where:

$$\mathbf{x} = \begin{bmatrix} \boldsymbol{\omega}_{BI} & \mathbf{a}_A \end{bmatrix}^T ; \; \mathbf{u} = \begin{bmatrix} L_C & M_C & N_C \end{bmatrix}^T \quad (137)$$

Observation. *For balance movement described above, where body frame coincide with reference frame, stability and command matrix are identically for attitude angles and rotation angles.*

Next we found an analytical solution of the equations. For this purpose we put matrix relations in scalar form.

From kinematical equation we obtain:

$$\Delta \dot{\phi} = \Delta \omega_x - \omega_j \Delta \psi ;$$
$$\Delta \dot{\theta} = \Delta \omega_y$$
$$\Delta \dot{\psi} = \Delta \omega_z + \omega_j \Delta \phi \quad (138)$$

From dynamic equations we can write:

$$\Delta \dot{\omega}_x = -a_x^x \omega_j \Delta \omega_x + a_x^z \omega_j \Delta \omega_z + 3 a_x^\phi \omega_j^2 \Delta \phi + b_x^l \Delta L_C + b_x^n \Delta N_C$$
$$\Delta \dot{\omega}_y = 3 a_y^\theta \omega_j^2 \Delta \theta + b_y^m \Delta M$$
$$\Delta \dot{\omega}_z = -a_z^x \omega_j \Delta \omega_x - a_z^z \omega_j \Delta \omega_z + 3 a_z^\phi \omega_j^2 \Delta \phi + b_z^l \Delta L_C + b_z^n \Delta N_C \quad (139)$$

where we denoted:

$$a_x^x = \frac{E(B - A - C)}{AC - E^2} ; \; a_x^z = \frac{CB - C^2 - E^2}{AC - E^2} ;$$

$$a_x^\phi = \frac{CB-C^2}{AC-E^2};\ a_y^\theta = \frac{A-C}{B};\ a_z^x = \frac{AB-A^2-E^2}{AC-E^2};$$
$$a_z^z = \frac{E(A+C-B)}{AC-E^2};\ a_z^\phi = \frac{E(B-C)}{AC-E^2};\ b_x^l = \frac{C}{AC-E^2};$$
$$b_x^n = \frac{E}{AC-E^2};\ b_z^n = \frac{A}{AC-E^2};\ b_z^l = \frac{E}{AC-E^2};$$
$$b_y^m = \frac{1}{B} \qquad (140)$$

Deriving equations (138) and substituting in (139) we obtain:

$$\Delta\ddot{\phi} + a_x^x\omega_j\Delta\dot{\phi} + (a_x^z - 3a_x^\phi)\omega_j^2\Delta\phi = (a_x^z-1)\omega_j\Delta\dot{\psi}$$
$$-a_x^x\omega_j^2\Delta\psi + b_x^l\Delta L + b_x^n\Delta N$$
$$\Delta\ddot{\theta} - 3a_y^\theta\omega_j^2\Delta\theta = b_y^M\Delta M$$
$$\Delta\ddot{\psi} + a_z^z\omega_j\Delta\dot{\psi} + a_z^x\omega_j^2\Delta\psi = (1-a_z^x)\omega_j\Delta\dot{\phi} +$$
$$(a_z^z + 3a_z^\phi)\omega_j^2\Delta\phi + b_z^l\Delta L_C + b_z^n\Delta N_C \qquad (141)$$

Similarly, for rotation matrix we obtain:

$$\Delta\ddot{\xi} + a_x^x\omega_j\Delta\dot{\xi} + (a_x^z - 3a_x^\xi)\omega_j^2\Delta\xi = (a_x^z-1)\omega_j\Delta\dot{\zeta}$$
$$-a_x^x\omega_j^2\Delta\zeta + b_x^l\Delta L + b_x^n\Delta N$$
$$\Delta\ddot{\eta} - 3a_y^\eta\omega_j^2\Delta\eta = b_y^M\Delta M$$
$$\Delta\ddot{\zeta} + a_z^z\omega_j\Delta\dot{\zeta} + a_z^x\omega_j^2\Delta\zeta = (1-a_z^x)\omega_j\Delta\dot{\xi} +$$
$$(a_z^z + 3a_z^\xi)\omega_j^2\Delta\xi + b_z^l\Delta L_C + b_z^n\Delta N_C \qquad (142)$$

First, we observe that the second equation can be analyzed separately.

For first and third equations, considering value of ω_J constant, we can apply Laplace transformation, and put these relations in matrix form:

$$\mathbf{A}(s)\mathbf{x} = \mathbf{bu} \qquad (143)$$

where:

$$\mathbf{A}(s) = \begin{bmatrix} s^2 + a_x^x\omega_j s + (a_x^z - 3a_x^\xi)\omega_j^2 & -(a_x^z-1)\omega_j s + a_x^x\omega_j^2 \\ (a_z^x-1)\omega_j s - (a_z^z + 3a_z^\xi)\omega_j^2 & s^2 + a_z^z\omega_j s + a_z^x\omega_j^2 \end{bmatrix} \qquad (144)$$

$$\mathbf{b} = \begin{bmatrix} b_x^l & b_x^n \\ b_z^l & b_z^n \end{bmatrix}\ \mathbf{x} = \begin{bmatrix}\xi & \zeta\end{bmatrix}^T;\ \mathbf{u} = \begin{bmatrix}L & N\end{bmatrix}^T \qquad (145)$$

Easily we can obtain inverse of $\mathbf{A}(s)$ matrix:

$$\mathbf{A}^{-1} = \frac{1}{P}\begin{bmatrix} s^2 + a_z^z\omega_j s + a_z^x\omega_j^2 & (a_x^z-1)\omega_j s - a_x^x\omega_j^2 \\ -(a_z^x-1)\omega_j s + (a_z^z + 3a_z^\xi)\omega_j^2 & s^2 + a_x^x\omega_j s + (a_x^z - 3a_x^\xi)\omega_j^2 \end{bmatrix} \qquad (146)$$

where characteristic polynomial is:

$$P(s) = s^4 + (a_x^x + a_z^z)\omega_j s^3 + (a_x^x a_z^z + a_z^x a_x^z + 1 - 3a_x^\xi)\omega_j^2 s^2 +$$
$$+(a_x^x + a_z^z - 3a_x^\xi a_z^z - 3a_z^\xi a_x^z + 3a_z^\xi)\omega_j^3 s +$$
$$+(a_x^x a_z^z + a_z^x a_x^z - 3a_x^\xi a_z^x + 3a_z^\xi a_x^x)\omega_j^4 \qquad (147)$$

Using these results we put previously relations in form

$$\mathbf{x} = \mathbf{A}^{-1}(s)\mathbf{bu} \qquad (148)$$

which represents analytical solution for commanded linear equations.

Observation. *For balance movement described above, the stability matrix* $\mathbf{A}$ *, defined by relation (144) are identically for attitude angles and rotation angles.*

VI. Extended satability and control matrices

Besides the general motion equations in linear form as outlined above, S/C needs other relationships to be added. Among them, the most important and which can not be neglected are the actuator equations and the guidance equations. For the autonomous flight, as is case of S/C ‘s, the guidance equation is necessary to introduce integrated terms specific to PID-type controllers.

For linearization to the Trigger Schmidt type command system,, we applied the method given by paper [7], using Fourier transform. Thus, by first harmonic approximation, we obtain a linear transfer function of the form:

$$N(s) \cong k_M^u \frac{s + \Omega_M}{s} \qquad (149)$$

where we denoted:

$$k_M^u = \frac{a_1}{x_0};\ \Omega_M = \frac{-b_1\omega}{a_1} \qquad (150)$$

where:

$$a_1 = \frac{2M}{\pi}\left(\sqrt{1 - \frac{(a_M + b_M)^2}{x_0^2}} + \sqrt{1 - \frac{a_M^{\ 2}}{x_0^2}}\right);$$
$$b_1 = -\frac{2M}{\pi}\frac{b_M}{x_0} \qquad (151)$$

where a_M, b_M, M define non linear function from figure 7, and sizes x_0 and ω means amplitude , respectively the pulsation of the input signal.

In this case, considering the integrator element and feedback loop, the linear transfer function of the command system for a channel is:

$$H_0(s) = \frac{k_M^u(s + \Omega_M)}{\tau_M s^2 + (k_M^u + 1)s + k_M^u\Omega_M} \qquad (152)$$

or, if you neglect the pulse term Ω_M , we obtain the following simplified linear relationship:

$$H_0(s) = \frac{k_M^u}{\tau_M s + k_M^u + 1} \quad (153)$$

Starting from the previously relation, the linear form of the command equation became:

$$\begin{aligned}&\left[\Delta\dot{L}_C \quad \Delta\dot{M}_C \quad \Delta\dot{N}_C\right]^T = \\ &= \mathbf{D}_M\left[\Delta L_C \quad \Delta M_C \quad \Delta N_C\right]^T + \mathbf{D}_u\Delta\mathbf{u}\end{aligned} \quad (154)$$

where:

$$\mathbf{D}_M = \begin{bmatrix} -\left(1+k_M^u\right)/\tau_M & 0 & 0 \\ 0 & -\left(1+k_M^u\right)/\tau_M & 0 \\ 0 & 0 & -\left(1+k_M^u\right)/\tau_{Mr} \end{bmatrix};$$

$$\mathbf{D}_u = \begin{bmatrix} k_M^u/\tau_M & 0 & 0 \\ 0 & k_M^u/\tau_M & 0 \\ 0 & 0 & k_M^u/\tau_M \end{bmatrix}, \quad (155)$$

Similarly, linear form of auxiliary equation (108) became:

$$\left[\Delta\dot{I}_\xi \quad \Delta\dot{I}_\eta \quad \Delta\dot{I}_\zeta\right]^T = \left[\Delta\xi \quad \Delta\eta \quad \Delta\zeta\right]^T \quad (156)$$

Using linear relation (154) and (156) we can build extended stability and control matrixes.

Table 3 extended stability matrix $\mathbf{A}$

	$\boldsymbol{\omega}_{BI}$	$\mathbf{a}_R$	$\mathbf{I}$	$\mathbf{M}_C$
$\boldsymbol{\omega}_{BI}$	$\mathbf{M}_\omega$	$\mathbf{M}_R$		$\mathbf{J}^{-1}$
$\mathbf{a}_R$	$\mathbf{W}_R$	$\mathbf{A}_R$		
$\mathbf{I}$		$\mathbf{I}_3$		
$\mathbf{M}_C$				$\mathbf{D}_M$

Table 4 extended control matrix **B**

	$u_l \quad u_m \quad u_n$
$\boldsymbol{\omega}_{BI}$	
$\mathbf{a}_R$	
$\mathbf{I}$	
$\mathbf{M}$	$\mathbf{D}_u$

Observation. *For balance movement described above, where body frame coincide with reference frame, extended stability and extended command matrix are identically for attitude angles and rotation angles.*

In this case the system can be put in standard form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (157)$$

where:

$$\mathbf{x} = \left[\boldsymbol{\omega}_{BI} \quad \mathbf{a}_R \quad \mathbf{I} \quad \mathbf{M}\right]^T;\ \mathbf{u} = \left[u_l \quad u_m \quad u_n\right]^T \quad (158)$$

VII. GUIDANCE COMMAND SYNTHESIS

A. Optimal control using uncoupled state vector

Resuming papers [4],[7] , the guidance commands for uncoupled state vector are the simple form:

$$\mathbf{u} = \mathbf{U}_R\left[u_\xi \quad u_\eta \quad u_\zeta\right]^T \quad , \quad (159)$$

where the main control signals are PID structure:

$$\begin{aligned} u_\xi &= -(k_u^\xi\tilde{\xi} + k_u^{\dot{\xi}}\dot{\tilde{\xi}} + k_u^{I\xi}\tilde{I}_\xi) \\ u_\eta &= -(k_u^\eta\tilde{\eta} + k_u^{\dot{\eta}}\dot{\tilde{\eta}} + k_u^{I_\eta}\tilde{I}_\eta) \\ u_\zeta &= -(k_u^\zeta\tilde{\psi} + k_u^{\dot{\zeta}}\dot{\tilde{\zeta}} + k_u^{I_\zeta}\tilde{I}_\zeta)\ , \end{aligned} \quad (160)$$

The matrix $\mathbf{U}_A$, were previously presented.

The parameters relative $\tilde{\xi};\tilde{\eta};\tilde{\zeta}$ are given by:

$$\tilde{\xi} = \xi - \xi_d;\ \tilde{\eta} = \eta - \eta_d;\ \tilde{\zeta} = \zeta - \zeta_d; \quad (161)$$

where ξ_d, η_d, ζ_d are input reference values, and the integrals term are defined hereby:

$$\begin{bmatrix}\dot{\tilde{I}}_\xi & \dot{\tilde{I}}_\eta & \dot{\tilde{I}}_\zeta\end{bmatrix} = \begin{bmatrix}\tilde{\xi} & \tilde{\eta} & \tilde{\zeta}\end{bmatrix}. \quad (162)$$

First, we will try to obtain a simplified solution for the guidance command defined previously in PID form. For this purpose we will start from scalar equations established for commanded linear equations:
Moreover we will neglect cross influence introduced by angular velocity ω_j and also we will considerate inertial product moment null:

$$E = 0$$

In this case all angular equations have a similar form:

$$\Delta\ddot{\xi} = \frac{\Delta L}{A};\ \Delta\ddot{\eta} = \frac{\Delta M}{B};\ \Delta\ddot{\zeta} = \frac{\Delta N}{C} \quad (163)$$

If we neglect actuator delay time

$$\tau_M = 0$$

from guidance command form established previously we can write following linear forms:

$$\Delta L = -(k_u^{\dot{\xi}}\Delta\dot{\tilde{\xi}} + k_u^{\xi}\Delta\tilde{\xi} + k_u^{I\xi}\Delta\tilde{I}_\xi)k_M^u/(k_M^u+1);$$
$$\Delta M = -(k_u^{\dot{\eta}}\Delta\dot{\tilde{\eta}} + k_u^{\eta}\Delta\tilde{\eta} + k_u^{I_\eta}\Delta\tilde{I}_\eta)k_M^u/(k_M^u+1);$$
$$\Delta N = -(k_u^{\dot{\zeta}}\Delta\dot{\tilde{\zeta}} + k_u^{\zeta}\Delta\tilde{\zeta} + k_u^{I_\zeta}\Delta\tilde{I}_\zeta)k_M^u/(k_M^u+1) \quad (164)$$

Separating angular inputs and applying Laplace transformation, from previously relation will obtain:

$$\left(A\frac{(k_M^u+1)}{k_M^u}s^2 + \frac{k_u^{\dot{\xi}}s^2 + k_u^{\xi}s + k_u^{I\xi}}{s}\right)\xi = \frac{k_u^{\dot{\xi}}s^2 + k_u^{\xi}s + k_u^{I\xi}}{s}\xi_D$$

$$\left(B\frac{(k_M^u+1)}{k_M^u}s^2 + \frac{k_u^{\dot{\eta}}s^2 + k_u^{\eta}s + k_u^{I\eta}}{s}\right)\eta = \frac{k_u^{\dot{\eta}}s^2 + k_u^{\eta}s + k_u^{I\eta}}{s}\eta_D$$

$$\left(C\frac{(k_M^u+1)}{k_M^u}s^2 + \frac{k_u^{\dot{\zeta}}s^2 + k_u^{\zeta}s + k_u^{I\zeta}}{s}\right)\zeta = \frac{k_u^{\dot{\zeta}}s^2 + k_u^{\zeta}s + k_u^{I\zeta}}{s}\zeta_D \quad (165)$$

Admitting proportionality between coefficients and inertial moment we can write:

$$k_1 = \frac{k_u^{\dot{\xi}}k_M^u}{A(k_M^u+1)} = \frac{k_u^{\dot{\eta}}k_M^u}{B(k_M^u+1)} = \frac{k_u^{\dot{\zeta}}k_M^u}{C(k_M^u+1)};$$
$$k_2 = \frac{k_u^{\xi}k_M^u}{A(k_M^u+1)} = \frac{k_u^{\eta}k_M^u}{B(k_M^u+1)} = \frac{k_u^{\zeta}k_M^u}{C(k_M^u+1)}$$
$$k_3 = \frac{k_u^{I\xi}k_M^u}{A(k_M^u+1)} = \frac{k_u^{I\eta}k_M^u}{B(k_M^u+1)} = \frac{k_u^{I\zeta}k_M^u}{C(k_M^u+1)} \quad (166)$$

Using these new coefficients, transfer function for angular size has the form:

$$H_0(s) = \frac{k_1 s^2 + k_2 s + k_3}{s^3 + k_1 s^2 + k_2 s + k_3}, \quad (167)$$

Next we use pole-zero allocation method [4]. For this purpose we use an optimal function quite similarly with the previously obtained:

$$H_0(s) = \frac{6.7\Omega_0 s^2 + 6.7\Omega_0^2 s + \Omega_0^3}{s^3 + 6.7\Omega_0 s^2 + 6.7\Omega_0^2 s + \Omega_0^3}, \quad (168)$$

with $\Omega_0 = \tau_r/t_r$, where $\tau_r = 1.5$, and response time is choose.
Identifying between functions coefficients, we obtain following useful relations:

$$k_1 = 6.7\Omega_0;\ k_2 = 6.7\Omega_0^2;\ k_3 = \Omega_0^3 \quad (169)$$

Finally, choosing response time $t_r = 5\ s$ we obtain:

$$k_1 = 2.01;\ k_2 = 0.603;\ k_3 = 0.027$$

B. Optimal control using coupled state vector

Supposing to have access to extend state vector $\mathbf{x}$, we can obtain directly the controller $\mathbf{K}$ for optimal command:

$$\mathbf{u} = -\mathbf{K}\mathbf{x} \quad (170)$$

In order to satisfy the linear quadratic performance index (cost function):

$$\min J = \int_0^\infty (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})\,\mathrm{d}t, \quad (171)$$

where the extended pair $(\mathbf{A},\mathbf{B})$ is controllable and the state weighting matrix $\mathbf{Q}$ is symmetric and quasi positive:

$$\mathbf{Q} \geq 0;\ \mathbf{Q} = \mathbf{Q}^T. \quad (172)$$

while the control weighting matrix $\mathbf{R}$ is symmetric and positive:

$$\mathbf{R} > 0;\ \mathbf{R} = \mathbf{R}^T; \quad (173)$$

In this case, the following relation gives the optimal controller

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} \quad (51)$$

where the matrix $\mathbf{P}$ is the solution of the algebraic Riccati equation:

$$\mathbf{A}^\mathbf{T}\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\mathbf{T}\mathbf{P} + \mathbf{Q} = \mathbf{0} \quad (174)$$

C. Optimal control using Kalman filter

Using the optimal controller designed above requires access to all system states, very difficult in view of the limited number of sensors. In this case, for a complete description of the system we use a linear state estimator constructed as a Kalman filter. For this purpose we start from the regular relations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbf{w}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \mathbf{v} \quad (175)$$

where **w** is the external noise and **v** is the internal noise introduced by the sensors, where the matrixes $\mathbf{G},\mathbf{C},\mathbf{D}$ are considerate corrected with the stability matrix with non-stationary variables $\mathbf{A}_1$

$$\mathbf{G} = [\mathbf{I} - \mathbf{A_1}]^{-1}\mathbf{G}_0 ; \mathbf{C} = [\mathbf{I} - \mathbf{A_1}]^{-1}\mathbf{C}_0 ; \mathbf{D} = [\mathbf{I} - \mathbf{A_1}]^{-1}\mathbf{D}_0 , \quad (176)$$

The idea of estimator operation is: if the deliver system $\Sigma_1 : (\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ with state **x**, can be "predicted" by system $\Sigma_2 : (\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ that uses state **z**, which is accessible in this case to be controlled. In order that the system Σ_2 follows the system Σ_1 we calculate a regulator **L** which brings the difference between actual read states $\mathbf{y_1}$ and estimated states $\mathbf{y_2}$ as a correction into the system Σ_2. In this case we can write:

$$\Sigma_1 : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{x}_0\delta + \mathbf{G}\mathbf{w} \\ \mathbf{y}_1 = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \mathbf{v} \end{cases} \quad (177)$$

$$\Sigma_2 : \begin{cases} \dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u} + \mathbf{z}_0\delta + \mathbf{L}(\mathbf{y}_1 - \mathbf{y}_2) \\ \mathbf{y}_2 = \mathbf{C}\mathbf{z} + \mathbf{D}\mathbf{u} \end{cases} \quad (178)$$

where initial conditions are introduced by $\mathbf{x}_0$, respectively $\mathbf{z}_0$. Tracking error, including the initial conditions, is given by:

$$\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{z} ; \quad \tilde{\mathbf{x}}_0 = \mathbf{x}_0 - \mathbf{z}_0 \quad (179)$$

If we decrease Σ_2 from Σ_1 and neglect the noise is obtained:

$$\tilde{\mathbf{x}} = he^{(\mathbf{A}-\mathbf{LC})t}\tilde{\mathbf{x}}_\mathbf{0} . \quad (180)$$

Hence if **L** is dimensioned such that **A-LC** has eigenvalues with negative real part, the estimation error tends to zero. Since **z** is provided by the estimator, we have access to all states to make control of the form:

$$\mathbf{u} = -\mathbf{K}\mathbf{z} \quad (181)$$

In this case the system Σ_1 is described by the equation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{z} + \mathbf{x_o}\delta = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}\tilde{\mathbf{x}} + \mathbf{x_o}\delta \quad (182)$$

which has the solution:

$$\mathbf{x} = he^{(\mathbf{A}-\mathbf{BK})t}(\mathbf{x_0}\delta + h\mathbf{B}\mathbf{K}e^{(\mathbf{A}-\mathbf{LC})t}\tilde{\mathbf{x}}_\mathbf{0}) \quad (183)$$

The process of calculating the estimator is similar to that described above for the optimal regulator. This is based on the dual system:

$$\dot{\overline{\mathbf{x}}} = \mathbf{A}^T\overline{\mathbf{x}} + \mathbf{C}^T\overline{\mathbf{u}} \quad (184)$$

for which is considered performance index:

$$\min J = \int_0^\infty [\overline{\mathbf{x}}'(\mathbf{G}\overline{\mathbf{Q}}\mathbf{G}^T)\overline{\mathbf{x}} + \overline{\mathbf{u}}^T\overline{\mathbf{P}}\overline{\mathbf{u}}]\,\mathrm{d}t \quad (185)$$

By solving the matrix Riccati equation:

$$\mathbf{A}\mathbf{R} + \mathbf{R}\mathbf{A}^T - \mathbf{R}\mathbf{C}^T\overline{\mathbf{P}}^{-1}\mathbf{C}\mathbf{R} + \mathbf{G}\overline{\mathbf{Q}}\mathbf{G}^T = \mathbf{0} \quad (186)$$

matrix estimator is obtained:

$$\mathbf{L} = \mathbf{R}\mathbf{C}^T\overline{\mathbf{P}}^{-1} \quad (187)$$

where **R** is the solution of Riccati equation.

VIII. Input Data, Calculus Algorithm and Results

A. Input data for the model

As input data for application we considered:
The eccentricity $e = 0.3$
The orbital period $T = 24\,h$
The inertial moments:
$A = 1[kgm^2]; B = 2[kgm^2]; C = 3[kgm^2]$
The product of inertia $E = 0.02[kgm^2]$
Parameters of the Schmidt Trigger element
$a_M = 0.1; b_M = 0.3;\ \tau_M = 0.1[s]; k_M^u = 2$.

B. Calculus algorithm

The calculus algorithm consists in multi-step method Adams' predictor-corrector with variable step integration method: [2] [16]. Absolute numerical error was 1.e-12, and relative error was 1.e-10.

C. Results

First we highlight the influence of gravitational moment on the uncontrolled satellite orientation. Figure 7 presents the rotational velocity around the y axis of the mobile frame related inertial frame

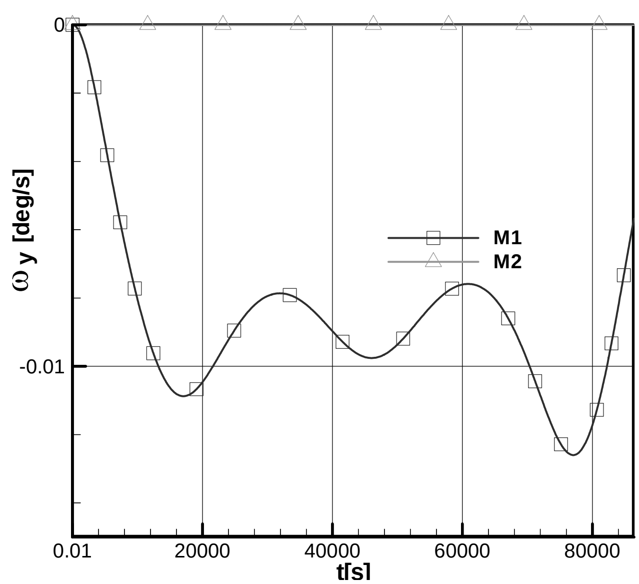

Fig. 7 Angular velocity for uncontrolled vehicle. M1 - with gravitational moment terms; M2- without gravitational moment terms

We can see that the gravitational influence leads to an additional angular velocity.

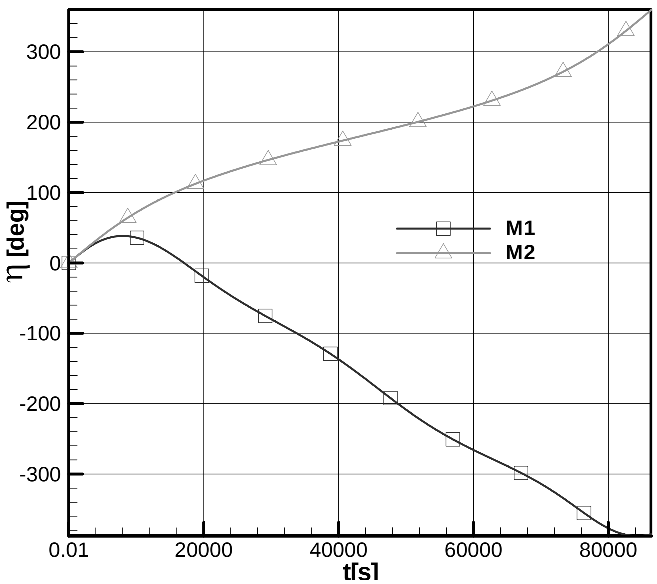

Fig. 8 angular diagram for uncontrolled vehicle. M1 - with gravitational moment terms; M2- without gravitational moment terms

Consequently, it influences the angle around the y axis, as we can see from Figure 8

Next we analyze the three types of orientation control systems described above. For starters, thrust control using a trigger Schmidt element is presented.

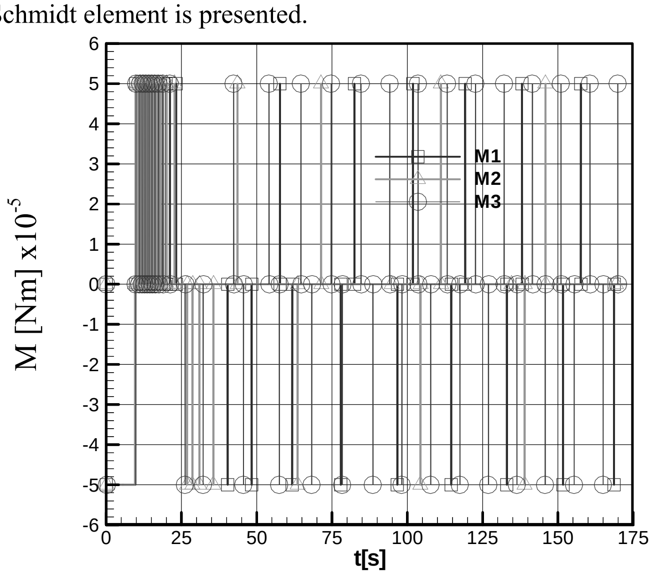

Fig. 8 command moment for controlled vehicle. M1- Optimal control using uncoupled state vector ; . M2- Optimal control using coupled state vector ;

Note that after achieving control system synthesis, the model uses nonlinear switching element. Because at the beginning we have an angular velocity jump, the command is more active in this moment. Applying the above presented control systems, the absolute angular velocity is stabilized at the base, which provides on the satellite a rotation velocity around its y axis synchronous with the motion around Earth, as we can see in figure 9.

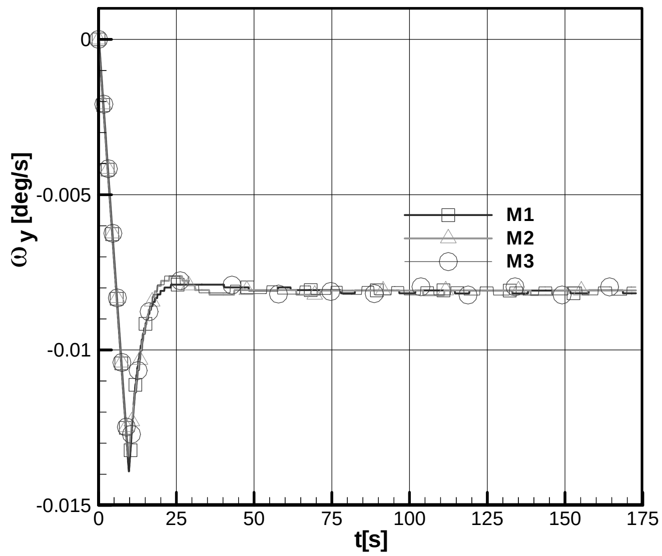

Fig. 9 angular velocity diagram for controlled vehicle. M1- Optimal control using uncoupled state vector ; . M2- Optimal control using coupled state vector ; M3 – Kalman filter;

Finally, figure 10 shows the rotation angle around the y axis, which is stabilized at null value, and providing the overlap of the mobile frame over reference frame.

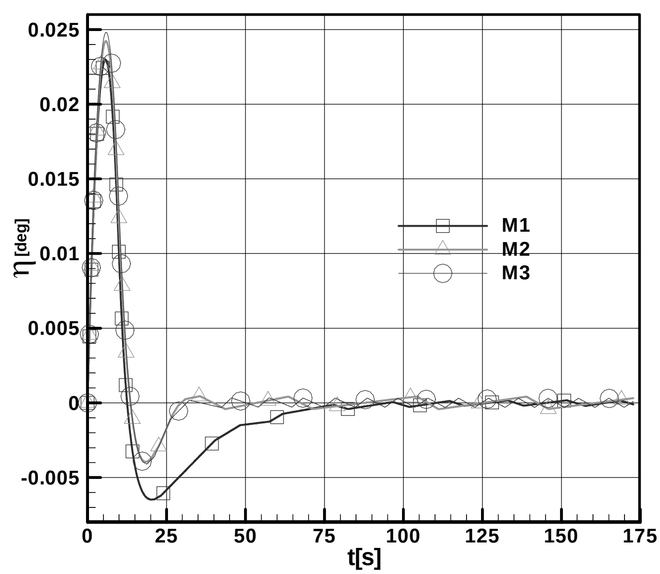

Fig. 10 angular diagram for controlled vehicle. M1- Optimal control using uncoupled state vector ; . M2- Optimal control using coupled state vector ; M3 – Kalman filter;

IX. Conclusions

The paper presents synthesis aspects of the simulation model, developed for the calculation of Attitude Control System-ACS of the small satellite which uses as command a micro jet engine. The application is made for three ACS variants first using control system for uncoupled state and the second using a control system for coupled state an third using Kalman filter . From the results obtained one can observe that the last two

solutions, although there are more complicated, are better than the previous ones, providing an ACS with the shortest response time and a smaller override.
As a general conclusion we must underline two novelty aspects introduced by the paper:
-We achieved the description of the model by using the rotation angles, which lead to polynomial forms for the rotation and connection matrix and which eliminate the singularities of the connection matrix in case of Euler's angles. On the other hand, these 3 values are independent and on the same time they have an angular dimension, and so they are measurable. This creates a great advantage on opposition to the usage of the Hamilton quaternion.
-By the linearization of the Trigger-Schmidt element we have constructed homogenous linear system and we made the ACS synthesis. With all the simplifications introduced by the Fourier transformation, the result obtained is valid, this being verified by testing the system in it's non-linear form.

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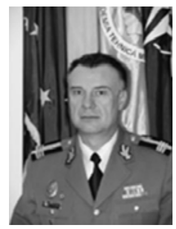

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