Investigation of the State Dependent Riccati Equation (SDRE) adaptive control advantages for controlling non-linear systems as a flexible rotatory beam[1]

Pierre Bigot[2] and Luiz C. G. de Souza[3]

***Abstract*—** Control of flexible structures is an open problem. Such structures can be very different, for example, robot arm or satellite solar panel. The common point between these structures is their very light weight and large length. Light structure control requires less energy, smaller actuators but a much more complex control system to deal with vibrations. In this paper a flexible rotatory beam is modeled by Euler-Bernoulli hypothesis and its angular position is controlled. This kind of model is, most of the time, highly non-linear. As a result, controller designed by linear control technique can have its performance and robustness degraded. To deal with this problem, the State-Dependent Riccati Equation (SDRE) method is used to design and test a position control algorithm for the rigid-flexible non-linear model. The Matlab/Simulink simulation is based on the characteristics of real flexible link equipment driven by a DC servomotor. This work serves to show the relevance this non-linear controller showing advantages it has over a more classic LQR controller. In future work, this controller will be tested with the real rotatory beam to validate the model and the SDRE control efficiency.

Keywords — *adaptative control, flexible , LQR , non-linear, rotatory beam, SDRE.*

I. INTRODUCTION

Even if the design of flexible Euler-Bernoulli is a well-known problem, it is still a subject of research [3, 13]. Moreover, most of the time, equations are linearized. Here, a first order non-linear kinematics is developed. The SDRE method [5] is an approach that can deal with non-linear plant; it linearizes the plant around the instantaneous point of operation and produces a constant state-space model of the system similar to LQR [6] control technique. The process is repeated in the next sampling periods therefore producing and controlling several state dependent linear models out of a non-linear one. In other words, SDRE controller is an adaptive LQR. For simplification, this work does not incorporate the Kalman filter technique; since it is assumed that all the states are known. Several simulations have proven the computationally feasibility for real time implementation [8].

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II. SDRE METHODOLOGY

Linear Quadratic Regulation (LQR) approach is well-known and its theory has been extended for the synthesis of non-linear control laws for non-linear systems [6]. This is the case for satellite dynamics that are inherently non-linear. Several methodologies exist for control design and synthesis of these highly non-linear systems; these techniques include a large number of linear design methodologies [10] such as Jacobian linearization and feedback linearization used in conjunction with gain scheduling [11]. Non-linear design techniques have also been proposed including dynamic inversion and sliding mode control [12], recursive back stepping and adaptive control [9].

Comparing with Multi-objective Optimization Non-linear control methods [2] the SDRE method has the advantage of avoiding intensive interaction calculation, resulting in simpler control algorithms more appropriated to be implemented in a satellite on-board computer.

The Non-linear Regulator problem [5] for a system represented by the SDRE form with infinite horizon, can be formulated minimizing the cost function given by

$$J(x_0,u)=\frac{1}{2}\int_{t_0}^{\infty}(x^TQ(x)x+u^TR(x)x)dt \qquad (1)$$

with the state $x\in\mathbb{R}^n$ and control $u\in\mathbb{R}^m$ subject to the non-linear system constraints given by

$$\begin{aligned}\dot{x}&=f(x)+B(x)u\\ y&=C(x)x+D(x)\mathrm{u}\\ x(0)&=x_0\end{aligned} \qquad (2)$$

where $B\in\mathbb{R}^{n\times m}$ and $C\in\mathbb{R}^{s\times n}$ are the system input and the output matrices respectively, and $x\in\mathbb{R}^s$ where s is the dimension of the output vector of the system. D is the feed forward matrix and will be considered null as in most of the systems there is no direct action of the control on the output. x_0 represents the initial conditions vector and $Q\in\mathbb{R}^{n\times n}$ and $R\in\mathbb{R}^{m\times m}$ are the weight matrices semi defined positive and defined positive respectively.

Applying a direct parameterization to transform the non-linear system into State Dependent Coefficients (SDC) representation [4], the dynamic equations of the system with control can be written in the form

$$\dot{x}=A(x)x+B(x)u \qquad (3)$$

with $f(x) = A(x)x$, where $A \in \mathbb{R}^{n \times n}$ is the states matrix. A is not unique. In fact there are an infinite number of parameterizations for SDC representation. There are at least two parameterizations A_1 and A_2 for all $0 \le \alpha \le 1$ satisfying

$$A(x) = \alpha A_1(x) + (1 - \alpha)A_2(x) \tag{4}$$

The choice of parameterizations must be made in accordance with the control system of interest. However, this choice should not violate the controllability of the system, i.e., the matrix controllability state dependent $[B(x) \quad A(x)B(x) \quad \dots \quad A(x)^{n-1}B(x)]$ must be full rank.

The State-Dependent Algebraic Riccati Equation (SDARE) can be obtained applying the conditions for optimality of the variation calculus. In order to simplify expressions, state dependent matrix are sometimes written without reference to the states x: i.e. $A(x) \equiv A$. As a result, the Hamiltonian for the optimal control problem Eq.(1) and Eq(2) is

$$H(x, u, \lambda) = \frac{1}{2}(x^T Q x + u^T R u) + \lambda(Ax + Bu) \tag{5}$$

where $\lambda \in \mathbb{R}^n$ is the Lagrange multiplier.

Applying to Eq.(5) the necessary conditions for the optimal control given by $\dot{\lambda} = -\frac{\partial H}{\partial x}$, $\dot{x} = -\frac{\partial H}{\partial \lambda}$ and $0 = -\frac{\partial H}{\partial x}$ leads to

$$\begin{aligned} -\dot{\lambda} &= -Qx - \frac{1}{2}x^T \frac{\partial Q}{\partial x} x - \frac{1}{2}u^T \frac{\partial R}{\partial x} u \\ &\quad - \left[\frac{\partial Ax}{\partial x}\right]^T \lambda - \left[\frac{\partial Bu}{\partial x}\right]^T \lambda \end{aligned} \tag{6}$$

$$\dot{x} = A(x)x + B(x)u \tag{7}$$

$$0 = R(x)u + B^T(x)\lambda \tag{8}$$

Assuming the co-state in the form $\lambda(x) = P(x)x$, which is dependent of the state, and using Eq.(8), the feedback control law is obtained as

$$u(x) = -R^{-1}(x)B^T(x)P(x)x \tag{9}$$

Substituting this results into Eq.(7) gives

$$\dot{x} = A(x)x - B(x)R^{-1}(x)B^T(x)P(x)x \tag{10}$$

To find the function P, $\lambda = P(x)x$ is differentiated with respect to the time along the path

$$\dot{\lambda} = \dot{P}x + PAx - PBR^{-1}B^T P \tag{11}$$

Substituting Eq.(11) in the first necessary condition of optimal control Eq.(6) and arranging the terms more appropriately results in

$$\begin{aligned} 0 &= \frac{1}{2}x^T \frac{\partial Q}{\partial x} x + \frac{1}{2}u^T \frac{\partial R}{\partial x} u \\ &\quad + x^T \left[\frac{\partial A}{\partial x}\right]^T Px + \left[\frac{\partial Bu}{\partial x}\right]^T Px \\ &\quad + (PA + A^T P - PBR^{-1}B^T P + Q)x \end{aligned} \tag{12}$$

Two important relations are obtained to satisfy the equality of Eq.(12). The first one is state-dependent algebraic Riccati equation (SDARE) which solution is $P(x)$ given by

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{13}$$

Once $P(x)$ is known, it is possible to know our controller K explicitely. The expression of our controller can be extracted from Eq.(9)

$$K = -R^{-1}(x)B^T(x)P(x) \tag{14}$$

The second one is the necessary condition of optimality which must be satisfied, it is given by

$$\begin{aligned} 0 &= \dot{P}x + \frac{1}{2}x^T \left[\frac{\partial Q}{\partial x}\right]^T + \frac{1}{2}u^T \left[\frac{\partial R}{\partial x}\right]^T u \\ &\quad + x^T \left[\frac{\partial A}{\partial x}\right]^T Px + \left[\frac{\partial Bu}{\partial x}\right]^T Px \end{aligned} \tag{15}$$

For some special cases, such as systems with little dependence on the state or with few state variables, Eq.(13) can be solved analytically. On the other hand, for more complex systems, the numerical solution can be obtained using an adequate sampling rate. It is assumed that the parameterization of the coefficients dependent on the state is chosen so that the pairs $\big(A(x), B(x)\big)$ and $\big(A(x), C(x)\big)$ are in the linear sense for every x belonging to the neighborhood about the origin, point to point, stabilizable and detectable, respectively. Then the SDRE non-linear regulator produces a closed loop solution that is locally asymptotically stable. An important factor of the SDRE method is that it does not cancel the benefits that result from the non-linearity of the dynamic system, because, it is not require inversion and no dynamic feedback linearization of the non-linear system.

III. Rotatory Beam's definition

A. Beam definition

Fig. 1 shows a representation of a flexible rotatory beam; it consists of a beam fixed to the rotor motor at one end and free at the other one. Euler-Bernoulli beam is used; this means that deformations are considered small. Parameters of the beam are the following: length L, linear density ρ, rigidity EI_z and the rotor motor parameters are: angular position $\theta(t)$, which is a rotation along the X-axis so gravity has no influence, inertia, J_m torque Γ_m and radius of the hub r. The beam displacement is $y(x,t)$ and the deflection angle is $\alpha(x,t)$. To simplify notation, y and α are used without referring to their variables and their partial derivative relative to the time t and the position x are respectively written $\dot{y}$ and y'.

It can be noted than the angular deflection (or beam slope) is related with de displacement according to:

$$\alpha(x,t) = \frac{\partial y(x,t)}{\partial x} = y' \tag{16}$$

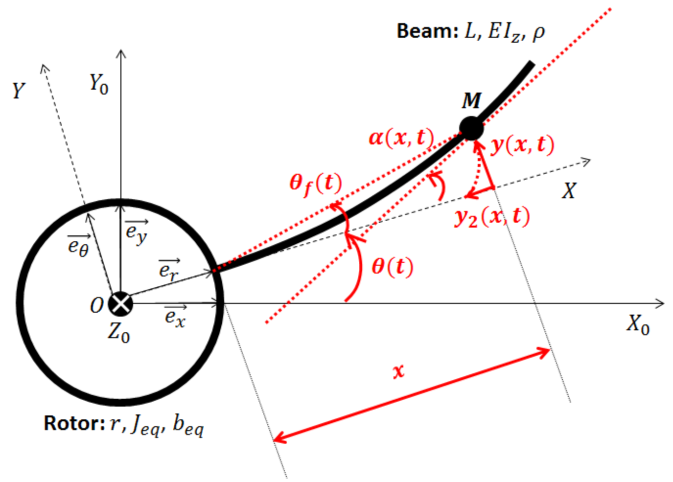

Fig. 1. Representation of the flexible beam

B. Kinematics

Let M be a point of the beam. In the beam frame $\mathcal{R}(X, Y, Z)$, coordinates of M are

$$OM = \begin{bmatrix} r + x \\ y \end{bmatrix}_{\mathcal{R}} \tag{17}$$

The velocity of this point is the derivative with respect to the inertial frame $\mathcal{R}_0(X, Y, Z)$. The beam is considered inextensible, so $\dot{x} = 0$.

$$v_M = \left.\frac{dOM}{dt}\right]_{\mathcal{R}_0} = \left.\frac{dOM}{dt}\right]_{\mathcal{R}} + \Omega_{\mathcal{R}/\mathcal{R}_0} \times OM|_{\mathcal{R}}$$
$$= \begin{bmatrix} -y\dot{\theta} \\ (r + x)\dot{\theta} + \dot{y} \\ 0 \end{bmatrix}_{\mathcal{R}} \tag{18}$$

C. Kinetic and potential energies

Kinetic energy of this system can be represented by two terms. The first one due to the motor rotation and the other due to the beam rigid-flexible motion

$$T_m = \frac{1}{2} J_m \dot{\theta}^2$$
$$T_b = \frac{1}{2} \rho \int_0^L v_M^T v_M \, dx \tag{19}$$

Second or more order in displacement y such as the axial displacement $y_2(x,t)$ won't be considered because of small deformations hypothesis. Then the total kinetic energy is

$$T = \frac{1}{2}\dot{\theta}^2 \left(J_m + \frac{1}{3}\rho((r + L)^3 - r^3) + \rho \int_0^L y^2 \, dx \right)$$
$$+ \frac{1}{2}\rho\dot{\theta} \int_0^L (r + x)\dot{y} \, dx + \frac{1}{2}\rho \int_0^L \dot{y}^2 \, dx \tag{20}$$

The potential energy of a flexible beam is given by

$$V = \frac{1}{2} EI_z \int_0^L y''^2 \, dx \tag{21}$$

In order to use energies to write the equations of motion, the beam deformation variable, that is, the displacement y, is need to be known explicitly. To do that, the assumed modes method is used.

D. Assumed modes

The motor rotation produces beam transverse vibrations. It is needed to analyze an infinitesimal element of the beam and consider moments and forces acting on it. Important elements of this analysis are shown in Fig. 2. Q is the shear force, M is the angular moment and ρ is the linear density.

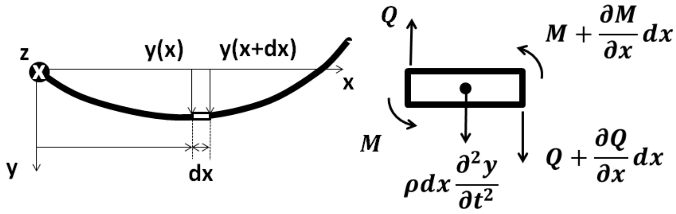

Fig. 2. Forces affecting the flexible beam

The application of the fundamental principle of the dynamics leads to Eq.(22) where the first one is the force in the direction of the axis y and the second one is the moment along the axis z.

$$\frac{\partial Q}{\partial x} = \rho \frac{\partial^2 y}{\partial t^2} \qquad Q = \frac{\partial M}{\partial x} \tag{22}$$

Moreover, for a prismatic beam, the rigidity EI_z is a constant so

$$M = -EI_z \frac{\partial^2 y}{\partial x^2} \tag{23}$$

Combining Eq.(22) and Eq.(23) leads to the general equation for transverse vibration of an uniform beam.

$$\frac{\partial^4 y}{\partial x^4} + \frac{\rho}{EI_z} \frac{\partial^2 y}{\partial t^2} = 0 \tag{24}$$

Looking for a solution for this equation as a product of temporal and spatial function of the form $y(x,t) = \Phi(x)q(t)$ given by

$$\Phi(x) = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x$$
$$q(t) = E\cos\omega t + F\sin\omega t \tag{25}$$
$$\beta^4 = \frac{\rho\omega^2}{EI_z}$$

Boundary conditions at beam ends are essential to determine the shape function Φ and parameters A, B, C and D. As the beam is clamped to the rotor, displacement and deflection are null at $x = 0$ (Eq.(26)) Likewise, the shear force and bending moment are zero at $x = L$ (Eq.(27)).

$$\Phi(0) = 0 \qquad \left.\frac{\partial \Phi}{\partial x}\right|_{x=0} = 0 \tag{26}$$

$$\left.\frac{\partial^3\Phi}{\partial x^3}\right|_{x=L} = 0 \qquad \left.\frac{\partial^2\Phi}{\partial x^2}\right|_{x=L} = 0 \tag{27}$$

Substituting Φ from Eq.(25) into Eq.(26) and Eq.(27) gives a system of four equations. This system can be easily reduced to one equation called characteristic equation.

$$\cos\beta L \cosh\beta L = -1 \tag{28}$$

The solution of Eq.(28) is an infinite set of spatial natural pulsations $\beta_i$where i is the mode number. The shape function Φ (or space Eigen function) from Eq.(25) associated to the mode i, called Φ_i, can now be written analytically. The first four mode shape are plotted in Fig. 3.

$$\begin{aligned}\Phi_i(x) &= A_i[\cosh\beta x - \cos\beta x \\ &\quad + k_i(\sinh\beta x - \sin\beta x)] \\ \text{with} \quad k_i &= \frac{\sin\beta L - \sinh\beta L}{\cos\beta L + \cosh\beta L}\end{aligned} \tag{29}$$

A finite number, of modes n is assumed to shape the beam deformation. The solution, Eq.(30), is a linear combination of all these modes:

$$y(x,t) = \sum_{i=1}^{n} \Phi_i(x) q_i(t) = \Phi^T q = q^T \Phi \tag{30}$$

Now that the displacement $y(x,t)$ is known explicitly, motion equations can be written using the Lagrange theory.

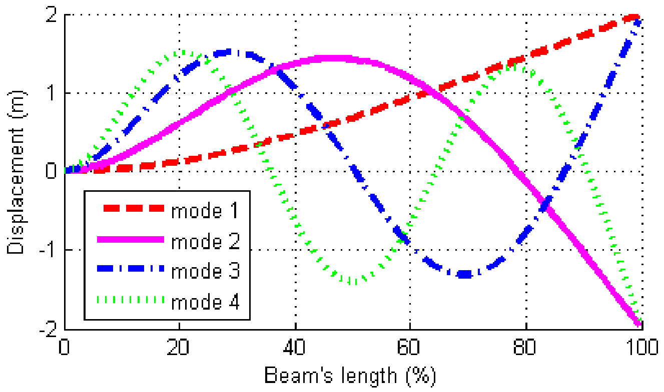

Fig. 3. First four shape functions $\Phi(x)$

E. Dynamic equation

In this section, Lagrange theory is used to develop motion equations. The generalized coordinates are the rigid motion, θ and the flexible modes q. External force F along the axis Y is considered null and along Z is equal to Γ_m, the rotor torque.

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{p}_i} - \frac{\partial T}{\partial p_i} + \frac{\partial V}{\partial p_i} \qquad \text{where } p_i = \{\theta, q\} \tag{31}$$

Substituting the expression of the displacement of Eq.(30) in kinetic and potential energies (Eq.(20) and Eq.(21) respectively)

$$\begin{aligned}T = \frac{1}{2}\dot{\theta}\left(J_m + \frac{1}{3}\rho((r+L)^3 - r^3) + \rho\int_0^L q^T\Phi\Phi^T q dx\right) \\ + \dot{\theta}\rho\int_0^L (r+x)\dot{q}^T\Phi dx \\ + \frac{1}{2}\rho\int_0^L \dot{q}^T\Phi\Phi^T\dot{q}dx\end{aligned} \tag{32}$$

$$V = \frac{1}{2}EI_z\int_0^L q^T\Phi''\Phi''^T qdx \tag{33}$$

and combining these results with Eq.(31) leads to the two Lagrange equations, according to the generalized coordinates θ and q:

$$\begin{aligned}\Gamma_m = \ddot{\theta}\left(J_m + \frac{1}{3}\rho((r+L)^3 - r^3) + \rho\int_0^L q^T\Phi\Phi^T q dx\right) \\ - 2\dot{\theta}\rho\int_0^L q^T\Phi\Phi^T\dot{q}dx \\ + \rho\int_0^L (r+x)\ddot{q}^T\Phi dx \\ 0 = \ddot{\theta}\rho\int_0^L (r+x)\Phi dx + \rho\int_0^L \Phi\Phi^T\ddot{q}dx \\ + \dot{\theta}^2\int_0^L \Phi\Phi^T q dx \\ + EI_z\int_0^L \Phi''\Phi''^T q dx\end{aligned} \tag{34}$$

that can be expressed on a matrix format

$$\begin{aligned}\begin{bmatrix}\Gamma_m \\ 0\end{bmatrix} = \begin{bmatrix}J_{eq} + q^T M_{ff} q & M_{rf}^T \\ M_{rf} & M_{ff}\end{bmatrix}\begin{bmatrix}\ddot{\theta} \\ \ddot{q}\end{bmatrix} + \begin{bmatrix}0 & 0 \\ 0 & K_{ff}\end{bmatrix}\begin{bmatrix}\theta \\ q\end{bmatrix} \\ + \begin{bmatrix}q^T M_{ff}\dot{q} & q^T M_{ff}\dot{\theta} \\ -M_{ff}q\dot{\theta} & 0\end{bmatrix}\begin{bmatrix}\dot{\theta} \\ \dot{q}\end{bmatrix}\end{aligned} \tag{35}$$

with the following parameters

$$\begin{aligned}J_{eq} &= J_m + \frac{1}{3}\rho((r+L)^3 - r^3) \\ M_{ff} &= \int_0^L \Phi\Phi^T dx \\ M_{rf} &= \rho\int_0^L (r+x)\Phi dx \\ K_{ff} &= EI_z\int_0^L \Phi''\Phi''^T dx\end{aligned} \tag{36}$$

Finally, some classical vibrating systems can be identified in Eq.(35): M the mass matrix, N_n the damping matrix, K the rigidity matrix and F the external force vector.

$$\begin{aligned}M &= \begin{bmatrix}J_{eq} + q^T M_{ff} q & M_{rf}^T \\ M_{rf} & M_{ff}\end{bmatrix} \qquad F = \begin{bmatrix}\Gamma_m \\ 0\end{bmatrix} \\ N_n &= \begin{bmatrix}q^T M_{ff}\dot{q} & q^T M_{ff}\dot{\theta} \\ -M_{ff}q\dot{\theta} & 0\end{bmatrix} \qquad K = \begin{bmatrix}0 & 0 \\ 0 & K_{ff}\end{bmatrix}\end{aligned} \tag{37}$$

F. DC motor

This system is controlled with the voltage delivered by the motor, thus, it is needed to express the motor torque Γ_m in function of the motor supply voltage U_m.

A classical model for a DC motor, taking losses into account, is giving in Eq.(38). The parameters involved in this equation are: the friction constant between the rotor and stator b_m the efficiency of the motor η_m, the efficiency of gears η_g, the motor torque constant K_t, the transmission constant of gears K_g, the back EMF constant K_m and the motor armature resistance R_m. All these parameters lead to the well-known motor equation

$$\Gamma_m = \frac{\eta_m \eta_g K_t K_g}{R_m}\left(U_m - K_g K_m \dot{\theta}\right) - b_m \dot{\theta} \tag{38}$$

Defining C_m as

$$C_m = \frac{\eta_m \eta_g K_t K_g}{R_m} \tag{39}$$

it is now possible to explicit the vector of external forces $F = [\Gamma_m \quad 0]^T$ as

$$F = \underbrace{\begin{bmatrix} C_m \\ 0 \end{bmatrix}}_{F_u} U_m - \underbrace{\begin{bmatrix} C_m K_g K_m + b_m & 0 \\ 0 & 0 \end{bmatrix}}_{N} \begin{bmatrix} \dot{\theta} \\ \dot{q} \end{bmatrix} \tag{40}$$

Substituting the result from Eq.(40) in Eq.(37) gives

$$M(q^2)\begin{bmatrix} \ddot{\theta} \\ \ddot{q} \end{bmatrix} + (N_n(q,\dot{q},\dot{\theta}) + N)\begin{bmatrix} \dot{\theta} \\ \dot{q} \end{bmatrix} + K\begin{bmatrix} \theta \\ q \end{bmatrix} = F_u U_m \tag{41}$$

Thus, in the global matrix equation of the system appears an additive damping term $N\dot{p}$.

It can be denoted than this system is not linear as the mass M and damping matrix N_n are not constants and depends on $q, \dot{q}$ and $\dot{\theta}$. According to small deformation hypothesis, the non-linear term $q^T M_{ff} q$ in the mass matrix M Eq.(37) is really small (order two in q), so it can be negligible. Therefore, since $\dot{\theta}$ and $\dot{q}$ are not necessarily small N_n terms Eq.(37) cannot be negligible [3].

To check the validity of these assumptions Fig.(4) represents the impulse response (1 second, amplitude 5V) to three different models: the linear model, the fully non-linear model and the non-linear model without non-linear term in the mass matrix M (partially non-linear). As it is possible to notify, fully and partially non-linear model have almost the same response whereas the linear model is quite different.

IV. SIMULATION STRATEGY

A. State Space Model

To be able to apply the SDRE technique, this system has to be represented using the state space model Eq.(3). States vector x and control u are defined as

$$x = [\theta \quad q \quad \dot{\theta} \quad \dot{q}]^T \qquad u = U_m \tag{42}$$

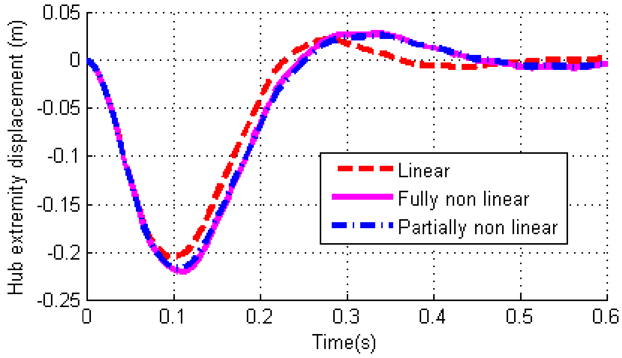

Fig. 4. Comparaison between different plant models

Reorganizing Eq.(41), the classic state space representation is obtained.

$$\dot{x} = \underbrace{\begin{bmatrix} 0_{n+1} & I_{n+1} \\ -M^{-1}K & -M^{-1}(N + N_n(x)) \end{bmatrix}}_{A(x)} x + \underbrace{\begin{bmatrix} 0_{n+1\times 1} \\ M^{-1}F_u \end{bmatrix}}_{B} U_m \tag{43}$$

with n, the number of flexible modes, and I, the identity matrix. Having a look at Eq.(43), matrix A depends on the state and matrix B is constant..

B. SDRE implementation

The algorithm is described in the Fig.(5). As the matrix A depends on the states it must be determined on every step. So, for every iteration of the simulation, states vector x is measured, the Riccati solution P is obtained from Eq.(11) the feedback control u is determined thanks to Eq.(15) and then, the new matrix A is obtained.

Implementation of this algorithm has been done using the MATLAB-Simulink. Riccati equation has been determinated via the S-function sfun_lqrysim [1] which permits working with real time simulation solving Riccati equations in Simulink without calling the Matlab interpreter. Fig.(6) represents the Simulink solution for the feedback control u.

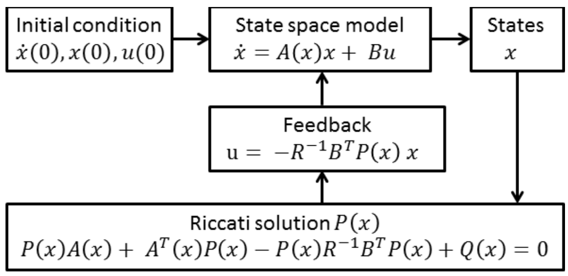

Fig. 5. SDRE algorithm

C. Performance requirements

Due to the physical features of the system, maximum voltage supply of the motor is 5V.

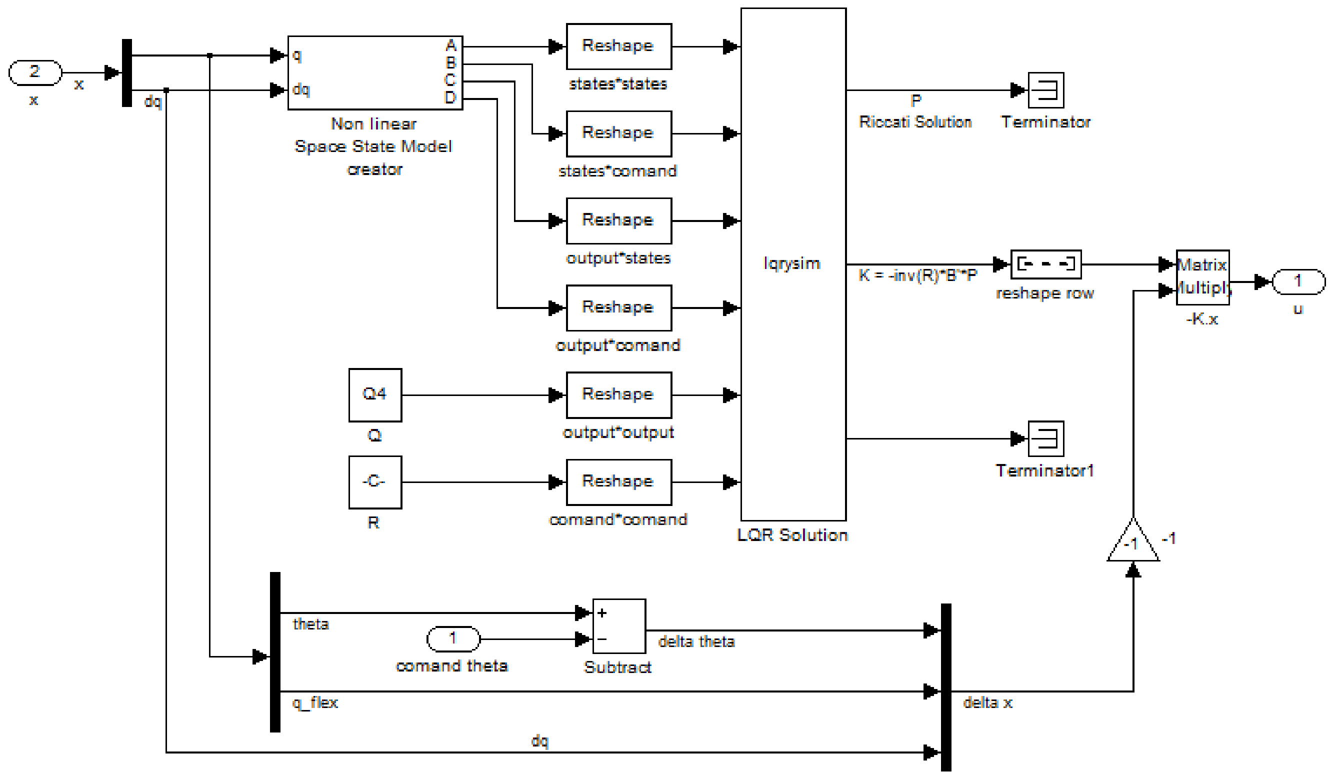

Fig. 6. Simulink feedback calculus with SDRE algorithm

Referring to performance objectives, those are temporal requirements since the model is non-linear and frequency analysis is not possible. Only one overshoot of the beam angular position θ is accepted, after what it shall be stabilized in the region $\pm 2\%$ in a minimum setting time T_s. To analyze the flexible motion the displacement at beam's extremity is measured. As small deformation hypothesis may not be infringed, maximum overshoot is ± 5 cm which corresponds to 10% of the beam length. Table I summarizes requirements defined in this section.

TABLE I. PERFORMANCE REQUIREMENTS

Parameter	Performance	Condition
θ	overshoot	Only one
θ	Rise time	Minimum
$y_L = y(x = L)$	Overshoot	≤ 5 [cm]
U_m	Voltage	≤ 5 [V]

V. SIMULATION RESULTS

TABLE II. MODEL PARAMETERS VALUES

Beam	Values	Motor	Values
L	48.26 cm	b_m	0.004 kg m^2 s^{-1}
EI_z	0.54 N m	J_m	0.002 kg m^2
ρ	0.1347 kg m^{-1}	C_m	0.1527 N V^{-1}

A. Model values

Table II shows the values used for the simulation.

B. Vibration mode analysis

To know the vibration frequencies, an eigenvalue analysis of the full system represented by Eq.(41) is done. For this analysis damping and external forces are considered null. Results are shown in table III.

TABLE III. VIBRATION FREQUENCIES

Mode n	1	2	3	4	5
f (Hz)	4.6	16.1	42.7	83.0	136.9

The estimation error for natural frequencies using Euler-Bernoulli theory is all the more important that the mode number is high [7]. That's why in this simulation, only the rigid mode and the first three flexible modes are taking into account.

C. Variation of parameters Q and R

SDRE technique is the generalized method of LQR for state dependent equation. In this paper, it is supposed that it is possible to get the SDRE solution P on-line (real time). If it was not possible, the solution would have been to calculate many SDRE point designs and to use scheduling, i.e. $P(x)$. In this study, LQR weights, R and Q, are chosen constant so they do not depend on the states. It means that the produced controller will depend on the states only because of non-linarites in A (N_n matrix, in Eq.(37), depend on $q, \dot{q}$ and $\dot{\theta}$).

The controller performance is directly related with these weights.

From the cost function represented by Eq.(1) it can be noted that matrix Q is linked with the states x and matrix R with the control signal u. In order to influence at each state separately Q matrix is chosen diagonal. Thus, each diagonal term is related to one state and acts as a penalty: the higher the value, less the influence of the state. The matrix R, (here a scalar because there is only one control variable) allows to size the control signal. In the same way as for Q, a high value of R penalizes the control signal. That is why, sometimes, Q can be called of performance weight and R energy weight.

In order to choose these parameters a set of Q and R values are tested. It has been notified during experiments that modifications on R influences almost only the control signal and not the dynamic response. For these reason, to determine Q and R values, first a set of Q values are tested; then, using the value that best matches the dynamical requirements, a set of R values are experimented to find the one which fulfilled the motor requirement: motor voltage $U_m \leq 5$ V.

From the results of this simulation it is possible to determine the influence of each term inside Q on the output. The first one related to θ reduces the setting time but increases overshoot in angle position and displacement. The second one, related to flexible states does not show a significant influence. The third one, related to the angular velocity helps to reduce slightly overshoot in θ. Finally, the last one related to the derivative of flexible states appears to reduce significantly displacement overshoot. Table IV represents Q values from which have been tried according to the logic described above to finally get Q_4; the best Q fulfilling our requirements Eq.(44).. The chosen value for R is 5.

TABLE IV. PARAMETERS Q TESTED

Q	θ	q	$\dot{\theta}$	$\dot{q}$
Q_1	1	1	1	1
Q_2	100	1	1	1
Q_3	100	1	5	320
Q_4	100	1	1	320

$$Q_4 = \mathrm{diag}(100,1,1,1,1,320,320,320) \quad R = 5 \tag{44}$$

Responses for a step input of 60° are shown in Fig. 7. Fig. 8 shows that the value of $R = 5$ enabled to fit the condition $U_m \leq 5$ V.

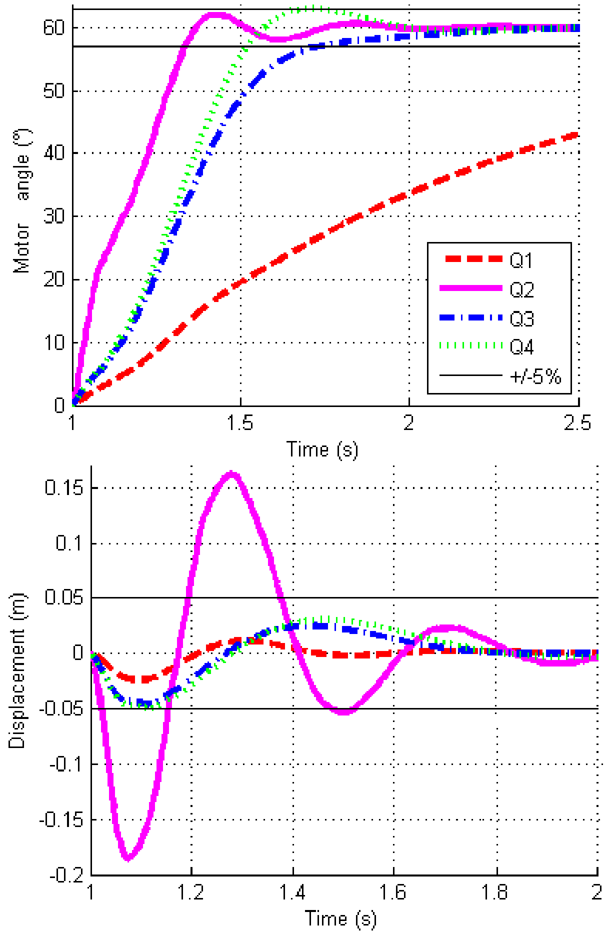

Fig. 7. Step response of the linear plant for differents Q

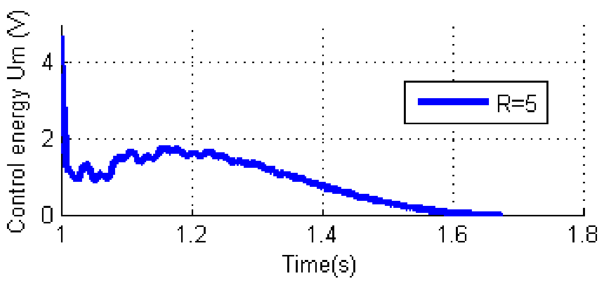

Fig. 8. Control U_m for $R = 5$

D. Comparaison with constant LQR controllers

To prove difficulties to control this system with a classic controller, constant LQR controllers have been tested for controlling the non-linear model in order to show advantages of an adaptive SDRE controller. Same weights as for the SDRE have been used to compare the behavior of our system controlled with LQR. Fig. 9 shows these results.

It can be seen clearly that these LQR controllers don't succeed to control the flexible beam. It shows that the flexible beam non-linearity is not negligible and plays an important part in the beam dynamic, and then an adaptive controller such as SDRE is necessary.

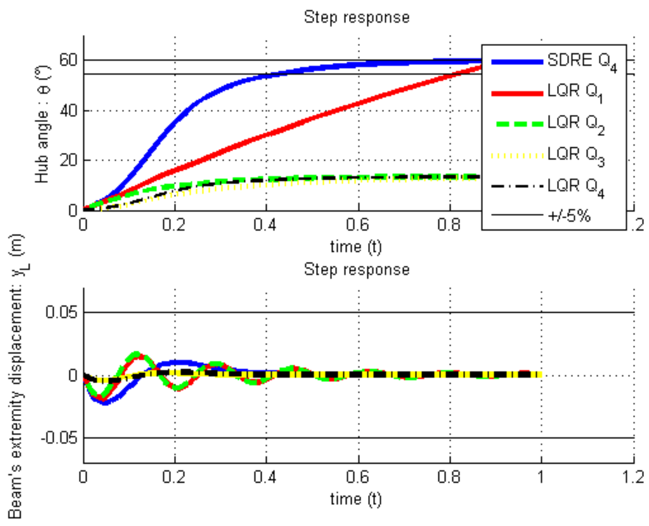

Fig. 9. Comparaison of some constants LQR with the best SDRE

VI. Conclusions

In this paper, the model of a rotatory Euler-Bernoulli beam is successfully built and the required performance objectives and physical requirements are achieved. This study shows how to implement a SDRE (State-Dependent Riccati Equation) controller for simulation. This controller model can be useful to control many other highly non-linear systems. The SDRE is tested here with a simple model for a flexible rotatory beam and shows great performances. Utility of the SDRE controller has been proved showing that a classic LQR controller can't control efficiently such non-linear systems. One of the main interests of this work is that, changing values of physical parameters such as beam length, or inertia, this model can easily be extended to many other systems such as satellite solar panels and antennas or robotic arm. In a future work this controller will be tested with the real rotatory beam in order to verify the real time implementation feasibility.

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