

Exploring Polar Equations With Interactive Spreadsheets

Andriy Didenko, David Allison, and Gary Miller

Abstract — Students who are studying calculus often find the graphing of polar equations difficult and time consuming. Microsoft Excel is a helpful software tool for the teaching of mathematics, and can be used to generate dynamic spreadsheets specifically designed to make mathematical tasks more instructive as well as visually stimulating for students.

A dynamic spreadsheet presentation can help students better appreciate how a graph is produced from the relationship between the polar coordinates.

Keywords—mathematics education technology polar equations Microsoft Excel spreadsheet

I. INTRODUCTION

Polar coordinates originated with the astronomical investigations of the early Islamic scholars [1], [2].

Polar coordinates are usually presented as a distance from a fixed point and an angle with a fixed direction. Alternatively, we could plot the radius versus the angle on two perpendicular axes, the way we do with rectangular coordinates. Adopting this approach, the graph of a circle would then appear as a straight line.

However, the standard representation of polar coordinates, as we know and use them today, originates in the seventeenth century, with the efforts of Isaac Newton's immediate predecessors. In fact Newton used the standard polar representation as one of nine different coordinate

Manuscript received June 20, 2009; Revised version received

The authors of this paper thank the Petroleum Institute for its support and financial assistance.

A. Didenko is in the Department of Mathematics, The College of Arts and Sciences, The Petroleum Institute, Abu Dhabi, The United Arab Emirates (phone: + 97 1 (0)2 607 5341; fax: +97 1 (0)2 607 5200; e-mail: adidenko@pi.ac.ae).

D. Allison is in the Department of Mathematics, The College of Arts and Sciences, The Petroleum Institute, Abu Dhabi, The United Arab Emirates (e-mail: dallison@pi.ac.ae).

G. Miller is in the Advancement University Placement Department, The Petroleum Institute, Abu Dhabi, The United Arab Emirates (e-mail: gmilller@pi.ac.ae).

systems in the Principia. A complete and fascinating history of the subject can be found in [3].

Before the digital era, a student's introduction to polar coordinates and polar graphs was necessarily restricted to a pencil and paper approach. This made the exploration of the subject matter somewhat tricky and error prone. However, the advent of information technology in the shape of the Internet, multimedia resources and educational software has supported and greatly enhanced the delivery of course materials at all educational levels, from kindergarten through to university [4-8]; although the impact of this technology has been most keenly felt within the higher education sector [9-11].

In a review paper [12] Baker & Sugden describe the variety of ways that spreadsheets have been useful in the teaching of various technical subjects. A number of books (see for example [13], [14]) have recently been published that describe how one can use Excel to perform mathematical modeling and charting.

In the teaching of mathematics the presentation of the concepts via equations, diagrams and tables is often static in style. The visual evidence available to students in support of the mathematical ideas is frequently restricted to a relatively small number of given textbook diagrams accompanying the text's examples. For students to generate additional graphical illustrations is time intensive. A consequence of this is that the more abstract topics of the subject have been difficult for instructors to explain to students. Arganbright [15], Wischniewsky [16], Abramovich & Sugden [17] and Wetzel & Whicker [18] have used Excel to make the teaching of mathematics more dynamic for an audience. El-Gebeily and Yushau [19], Didenko, Allison & Miller [20-28] have employed animated spreadsheet presentations to help students understand college level calculus topics.

Another issue is that of student motivation. The emergence of digital math presentation tools has made it easier to engage and increase the appreciation of students, especially those who are not majoring in mathematics. Prior to the information technology era, the static imagery coupled with the pencil and paper computations,

contributed to a larger number of students becoming frustrated and bored. Also, it was not a simple task to satisfy the audience's quite reasonable demand for real-world applications of the mathematical topics to be demonstrated.

A particular topic that arises in the traditional college calculus sequence is the polar coordinate system, and the affiliated task of graphing a polar equation [27]. A dynamic spreadsheet presentation can help the student appreciate how a curve is generated from the relationship between the polar coordinates.

Students who are non-math majors taking a standard calculus I-II-III sequence, sometimes find it difficult to make the transition from Cartesian coordinates to polar coordinates. In particular, the relationship between a curve and a polar equation $r = f(\theta)$ has aspects that may elude the neophyte. For instance the importance identifying values of θ corresponding to $r = 0$ or to where r attains an extreme value may not be sufficiently appreciated. An additional subtlety that can be missed is the range of θ -values that produces one copy of the polar curve.

Prior to age of the graphing calculator and the computer algebra system, the student's introduction to the sketching of polar curves typically involved a sequence of cases:

- a) The simple graphs $r = \text{constant}$ (circle) , $\theta = \text{constant}$ (line)
- b) A graph such as $r = 2 \cos \theta$ (circle), produced by the student laboriously computing a table of (r, θ) pairs by 'hand' and then plotting the points that arise
- c) Sketching a curve such as $r = 1 - 2 \sin \theta$ (Limacon with a loop) by: drawing the graph of $r = 1 - 2 \sin \theta$ in Cartesian coordinates, and then using this to read off the r -values that correspond to increasing values of θ . Plenty of opportunities for the unwary to err.

The advent of sophisticated hand-held and desktop educational technology has greatly facilitated the exploration of polar graphing: taking the tedium out of the time intensive generation of (r, θ) tables; allowing the student to rapidly produce graphs of stimulating and sometimes mesmerizing complexity. That having been said however, there are many schools and colleges whose instructors and students do not have access to state of the art math software tools such as Mathematica, or to high powered graphing calculators like the Texas Instruments TI-89. Nonetheless, there are numerous such institutions and individuals who do have access to a PC loaded with a version of Microsoft Office. The Excel component of this suite is a useful instructional aid for the teaching of higher level mathematics [28], [29], [30]. In this paper the authors

highlight the ability of Excel to effectively and rapidly produce curves of polar equations.

II. EXCEL AS A POLAR GRAPHING TOOL

In this section of the paper the authors describe an in-house developed Excel spreadsheet that can serve as a graph generator for polar equations. The spreadsheet is a cost-effective user friendly teaching tool.

The authors wish to emphasize that the development of the spreadsheet did not require any software programming skills (in Visual Basic for example). They set out to develop a tool that is easy to construct and utilize. Only certain basic features of the Excel package were needed, such as cell referencing and table calculations. Key cells for data entry were designed to incorporate pop-up comments / instructions to aid the user.

The spreadsheet features a list of pre-set polar functions of the form $r = f(\theta)$, typical of cases found in most standard calculus sequence textbooks [31]. These are accessible via a pull-down menu, and examples are shown below.

FIGURE 1 shows the four leaf polar curve corresponding to the selection $r = 3a \cos(2t)$ where the variable t is used in the Excel spreadsheet in place of the polar angle θ , and the coefficient a is an additional parameter appearing in the equation. The angle t and the parameter a can be individually varied by the user by means of Excel 'slider' control bars.



As the value of the parameter t is altered, the point corresponding to the t -value moves along the curve.

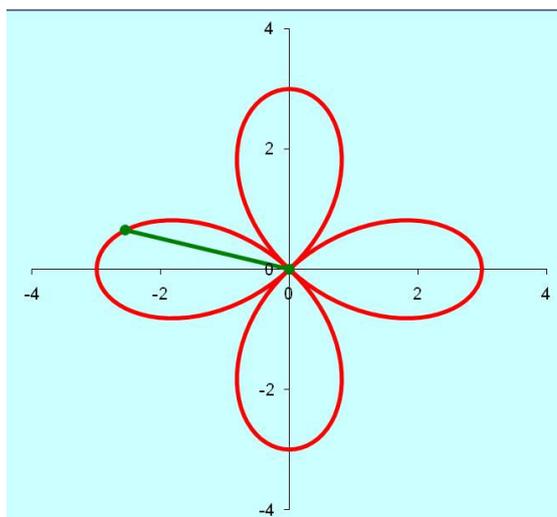


Fig. 1

The slider bar is a useful feature since it readily allows the student to explore functional aspects such as: the particular values of the polar angle that correspond to a zero value for the radius, and the ranges of the angle that trace the polar curve.

The spreadsheet can deal with more complicated polar graphs, as illustrated in FIGURE 2. This displays the graphical output corresponding to the case

$$r = a \sin(t) + a \sin^3\left(\frac{5t}{2}\right)$$

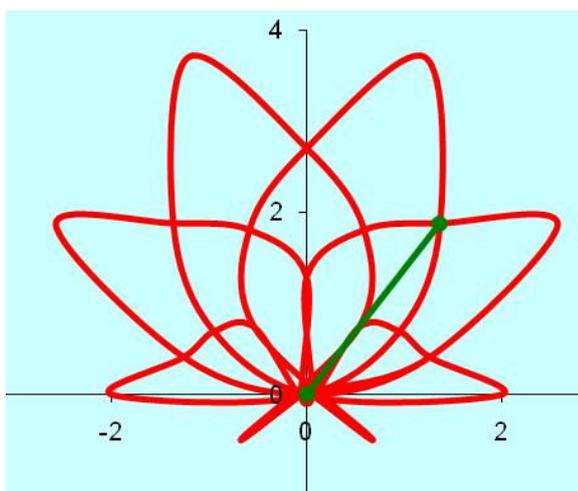


Fig 2.

When the user selects the polar equation

$$r = e^{\sin t} - 2 \cos(4t)$$

the rather striking image of a butterfly is created (see FIGURE 3).

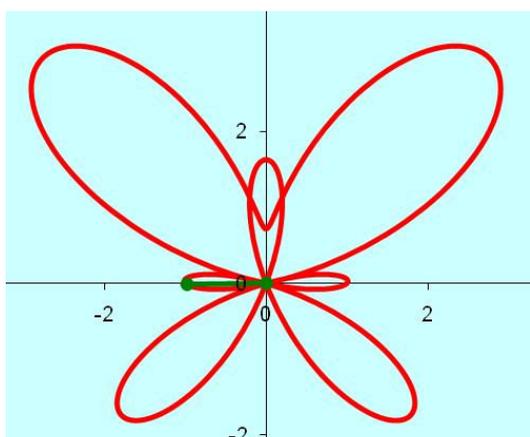


Fig. 3

III. DATASET DESCRIPTION

The dataset in the spreadsheet is organized in the following way.

E1:I26 is a table that contains the list of pre-set polar functions in the text and Excel-style formats as well as endpoints of the interval for the independent variable and the reference number of the function. The reference number is also used to select a function from a drop-down list (selector - combo box). In particular, E2:E26 is a list of pre-defined polar functions to be displayed in the selector-combo box and G2:G26 is the ordered list of the descriptive formulas for the polar functions used in the calculation of the values of r in B3:B103.

A2:B103 is a one-variable data table where the range A2:A103 contains values of θ between α and β with the step $(\beta - \alpha)/n$, and the range B3:B103 contains corresponding values of r calculated according to the formula stored in the cell B2. When a function is selected from the drop-down list, MS Excel function INDEX helps to identify values of α and β from the table. The values of α and β are used to generate the set of values for the independent variable that are stored in the A2:A103 array. Simultaneously, the reference number of the function allows one to choose a proper Excel-style formula to be stored in the cell B2 of the one-variable data table. Actually, the cell B2 contains a look-up function that selects a polar formula from a pre-defined list depending on the choice made by the user.

C3:D103 contains x - and y -values of points forming the curve. These values are calculated by the usual formulas $x = r \cos t$, $y = r \sin t$.

IV. FURTHER GRAPHICAL EXPLORATIONS

A spiral shaped curve called a "lituus" appears in FIGURE 4. For some interesting historical background to this curve see reference [32]. Mathematically [33], the word lituus refers to a curve for which the polar equation is given by

$$r^2 \theta = k$$

where k is a positive constant. The spiral has a pair of branches depending on the sign of r and is asymptotic to the x -axis. The points of inflection of the graph are located at

$$(r, \theta) = \left(\sqrt{2k}, \frac{1}{2} \right)$$

and

$$(r, \theta) = \left(-\sqrt{2k}, -\frac{1}{2}\right)$$

The equation for the graph in FIGURE 4 is

$$r = \frac{1}{\sqrt{t}}$$

where, in this particular case, $k = 1$ and the spreadsheet variable t is used in place of the polar angle θ .

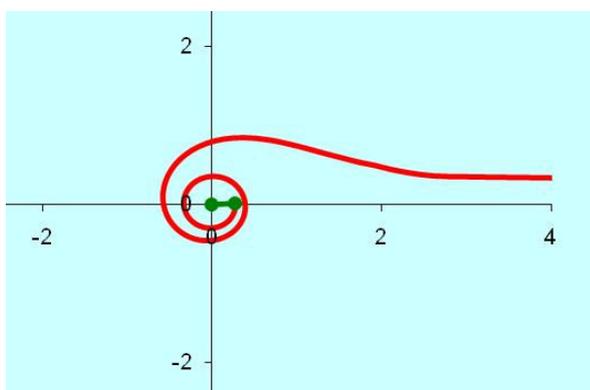


Fig. 4

Students are surprised that the graph corresponding to the linear equation

$$r = t$$

is also a spiral (see FIGURE 5). They feel that the result runs counter to their not unreasonable expectations that the output should be a straight line.

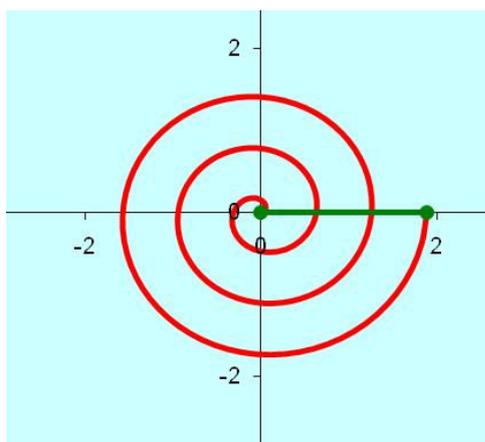


Fig. 5

FIGURE 6 is the peanut-shaped curve associated with the polar equation

$$r = \sqrt{1 - 0.8 \sin^2 t}$$

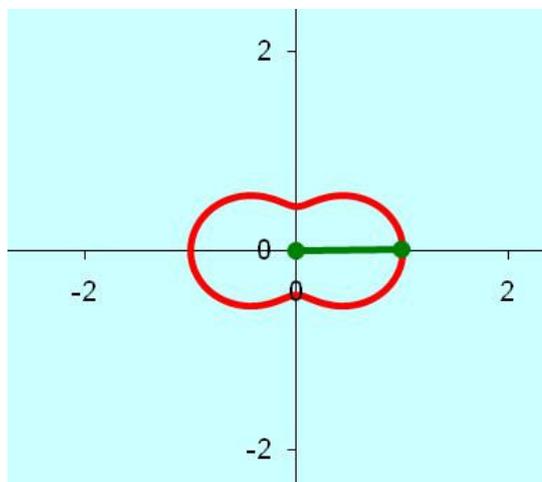


Fig. 6

FIGURE 7 shows the propeller shaped curve that arises when the polar equation

$$r = 2 + \sin(3t)$$

is graphed by the spreadsheet. Again, students tend to be somewhat surprised by the output. They expect to encounter a graph that is more overtly sinusoidal in character. The practice of graphing polar equations throws up many such cases, and it is only after a long period of working with polar coordinates and generating polar graphs that a student comes to feel a reasonable degree of confidence in being able to sketch by hand a curve that illustrates the relationship between polar variables for a given equation.

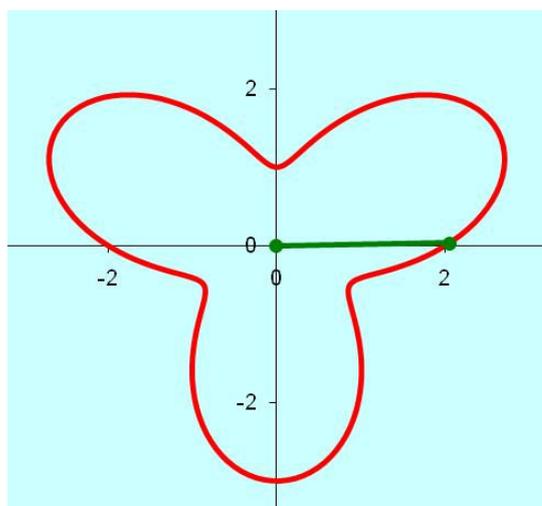


Fig. 7

A standard exercise appearing in many textbooks is the one of graphing the equation

$$r = 1 + 2 \cos(t)$$

The graph for the equation is shown in FIGURE 8

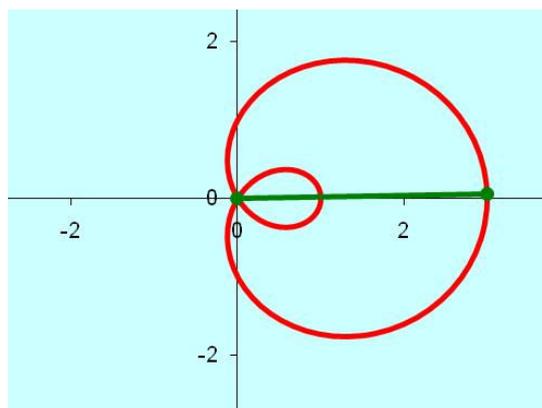


Fig. 8

Once the students have performed the graphing procedures connected to the polar equations related to FIGURES 4 – 8, it is often instructive for the teacher to ask the students to sketch just using pencil and paper the graphs corresponding to the equations:

$$r = 1 + 2 \sin(t)$$

$$r = 2 + \cos(3t)$$

prior to checking their results against the spreadsheet's outputs.

Next consider the graph appearing in FIGURE 9. The effect of scaling the independent parameter by a third in a cosine is shown.

$$r = \cos\left(\frac{t}{3}\right)$$

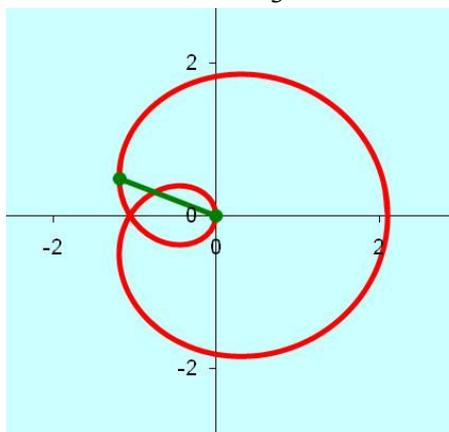


Fig. 9

Below (FIGURE 10.) is the graph of the straight line whose polar equation is

$$r = \frac{2}{2 \sin t - \cos t}$$

The corresponding equation in rectangular coordinates is

$$y = \frac{x}{2} + 1$$

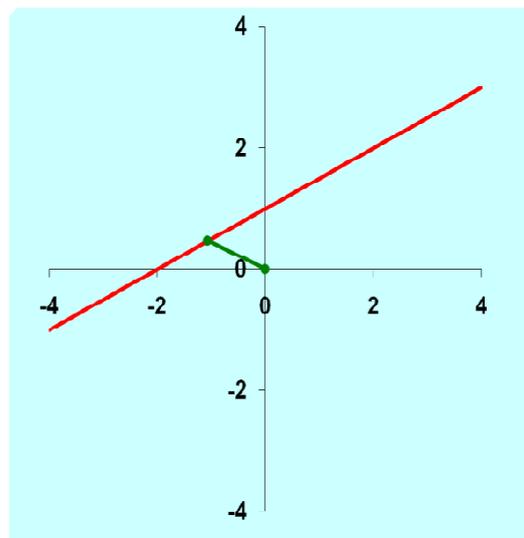


Fig. 10

At the present time the spreadsheet described in this paper is restricted in not being able to generate straight line graphs corresponding to polar equations of the form

$$t = C$$

where C is a constant.

The series of floral patterns (FIGURES 11-14) are generated from polar equations possessing the form

$$r = \sin(nt) \quad , \quad 0 \leq t \leq 2\pi$$

where n is a constant and is associated with the multiplicity of the polar angle. The graphical output clearly demonstrates to the student the property that as the multiplicity increases the number of leaves on the curve also increases. What is also illustrated is the interesting property that there are n leaves present when n is an odd integer, and $2n$ leaves appear when the multiplicity is even.

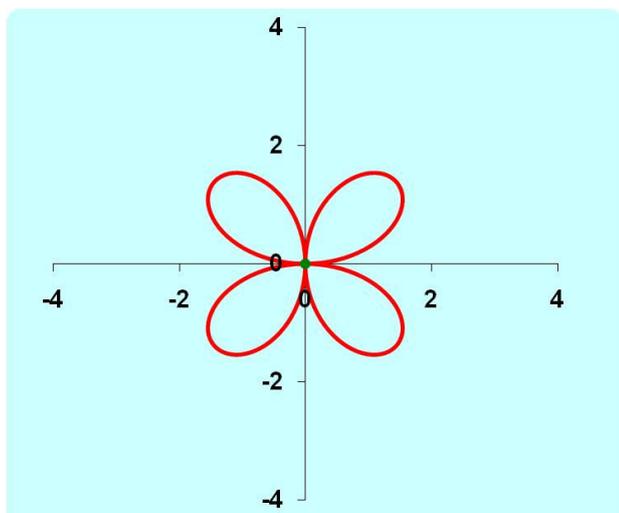


Fig. 11 (n=2)

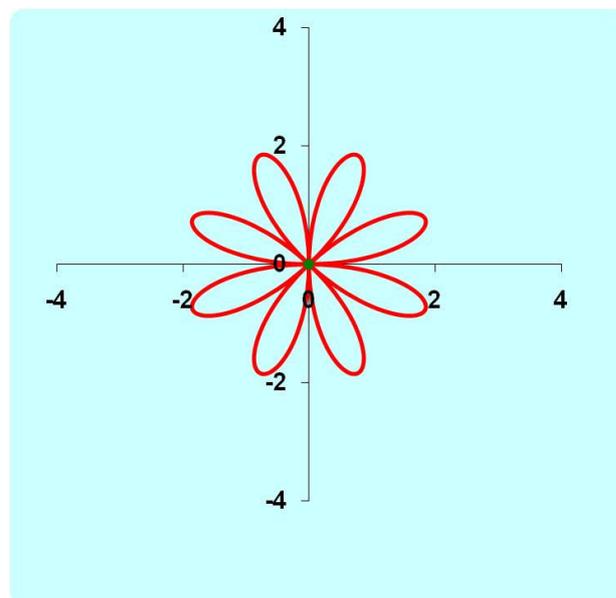


Fig. 13 (n=4)

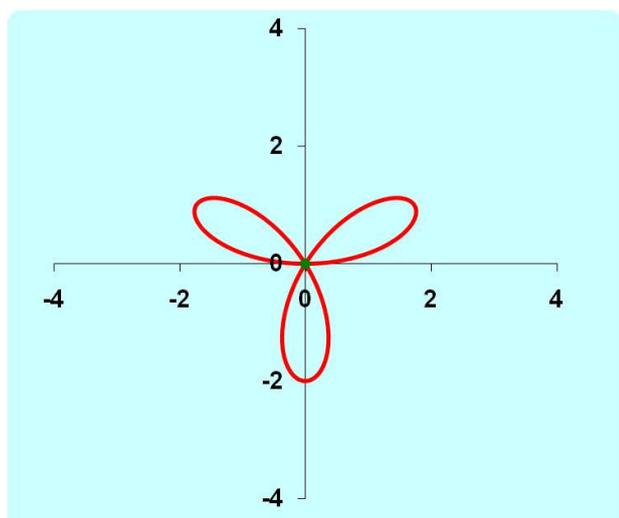


Fig. 12 (n=3)

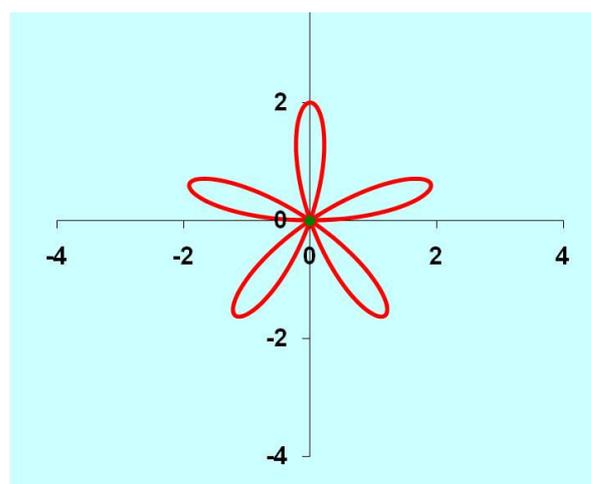


Fig. 14 (n=5)

Students are generally familiar with the equation of a circle expressed in Cartesian coordinates. This takes the form of an implicit equation. For example, a family of circles centered at the origin with radius a is described by the implicit equation

$$x^2 + y^2 = a^2$$

However, by changing to polar coordinates the equation attains the simpler form

$$r = a$$

In effect, by the use of polar coordinates one has parametrized the curves in a thoroughly ‘natural’ way.

FIGURE 15. shows the spreadsheet’s graphical output for the case when $r = a$. The value of a can be altered by means of a slider control.

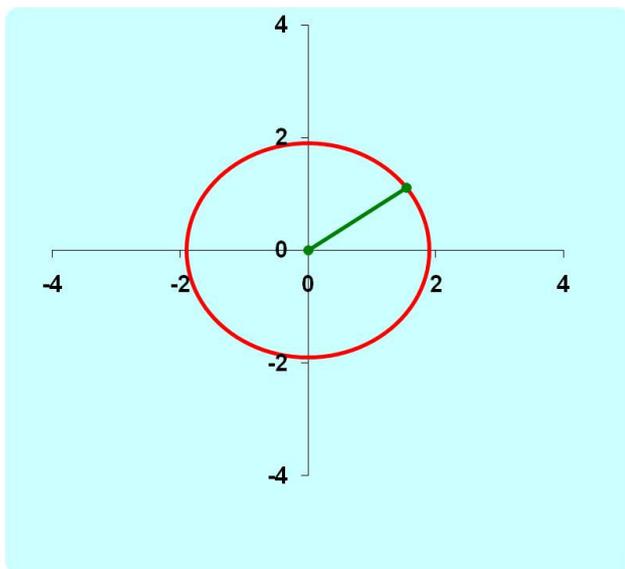


Fig. 15

A circle whose equation depends upon the polar angle is shown in FIGURE 16. The equation is

$$r = a \cos t$$

where again the spreadsheet allows one to change the value of the parameter a using a slider.

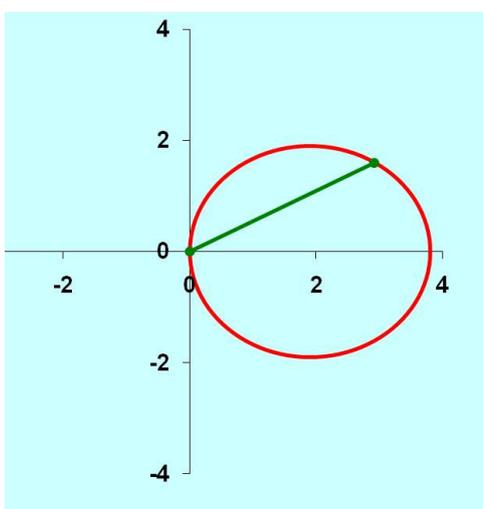


Fig. 16

The final diagram (FIGURE 17.) corresponds to the circle having a polar equation

$$r = a \sin t$$

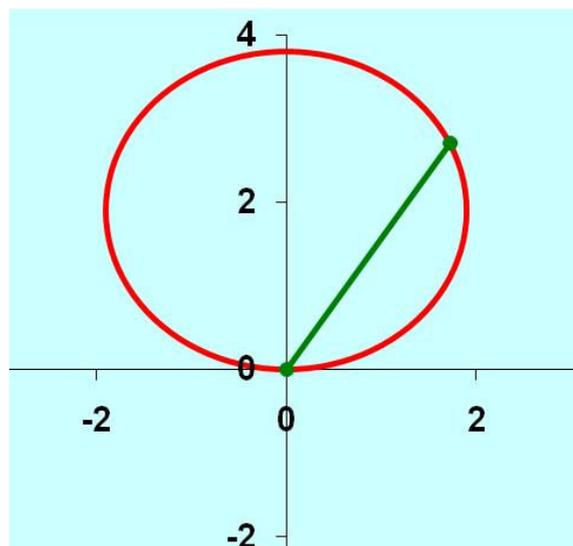


Fig. 17

V. CONCLUSION

The authors of this paper have incorporated the spreadsheet in their own teaching of the topic of polar coordinates to undergraduate engineering students. At the expository stage, before the students individually work through set calculation problems, they watch an animated spreadsheet presentation. Subsequently, the students can access the software to further explore the mathematical topic at their own pace and at a time convenient to them. A poll of the students revealed that they found the software to be simple to use and helpful to them in clarifying the nature of polar coordinates.

ACKNOWLEDGMENT

A.Didenko, D. Allison and G. Miller wish to express thanks to the students of the Petroleum Institute in Abu Dhabi who took part in the calculus classes during which the polar equation spreadsheet was utilized. The students’ input as to the effectiveness of the software as a pedagogic tool has been very useful. Additional thanks are conveyed to those teaching colleagues of the authors at the Petroleum Institute who generously gave up time to ‘test drive’ the software and record their observations.

REFERENCES

- [1] O'Connor, John J.; Robertson, Edmund F., "Abu Arrayhan Muhammad ibn Ahmad al-Biruni", *MacTutor History of Mathematics archive*.
- [2] T. Koetsier, L. Bergmans (2005), *Mathematics and the Divine*, Elsevier, p. 169, ISBN 0444503285
- [3] Coolidge, Julian (1952). "The Origin of Polar Coordinates". *American Mathematical Monthly* **59**: 78–85. doi:10.2307/2307104. http://www-history.mcs.st-and.ac.uk/Extras/Coolidge_Polars.html.
- [4] R.M. Reis, Development of Educational Software, *International Journal of Education and Information Technologies*, Vol. 1, Iss. 3 [2007], pp. 167-171.
- [5] R. Fang, H.J. Yang, H.L. Tsai, C.J. Lee and D. Li, The Digital Instrument Support Teacher's Instructional Innovation , *International Journal of Education and Information Technologies*, Vol. 1, Iss. 4 [2007], pp. 215-222.
- [6] R. Fang, H.J. Yang, H.L. Tsai, C.J. Lee, T. Tsai and D. Li, A Type of Technological Course with using Information Technology, *International Journal of Education and Information Technologies*, Vol. 1, Iss. 4 [2007], pp. 181-187.
- [7] I.I. Siller-Alcala, R. Alcantara-Ramirez, J. Jaimes-Ponce and J.P. Vaca-Pliego, How to link theory and practice of control engineering: Practices in real time and simulation, *International Journal of Education and Information Technologies*, Vol. 1, Iss. 2 [2007], pp. 114-120.
- [8] O.V. Vejar, Implementation of Schooling Based on Technologies for Communication and Information in the Subject Physics II, *Distance Learning, Multimedia and Video Technologies, Proceedings of the 8th WSEAS International Conference on Multimedia, Internet and Video Technologies/8th WSEAS International Conference on Distance Learning and Web Engineering, University of Cantabria, Santander, Spain, 2008*, pp. 61-66.
- [9] H. Z. Poor, Administration of Faculties by Information and Communication Technology and Its Obstacles, *International Journal of Education and Information Technologies*, Vol. 2, Iss. 1 [2008], pp. 24-30.
- [10] R. Matusu, J. Vojtesek and T. Dulik, Technology-Enhanced Learning Tools in European Higher Education, *Distance Learning, Multimedia and Video Technologies, Proceedings of the 8th WSEAS International Conference on Multimedia, Internet and Video Technologies/8th WSEAS International Conference on Distance Learning and Web Engineering, University of Cantabria, Santander, Spain, 2008*, pp. 51-54.
- [11] H.M. El-Bakry and N. Mastorakis, Realization of E-University for Distance Learning, *Distance Learning, Multimedia and Video Technologies, Proceedings of the 8th WSEAS International Conference on Multimedia, Internet and Video Technologies/8th WSEAS International Conference on Distance Learning and Web Engineering, University of Cantabria, Santander, Spain, 2008*, pp. 17-31.
- [12] J. Baker and S. Sugden (2003), Spreadsheets in Education – The First 25 Years, Spreadsheets in Education (eJSiE): Vol. 1: Iss. 1, Article 2
- [13] E. Neuwirth and D. Arganbright (2004), The Active Modeler: Mathematical Modeling with Microsoft Excel. Thompson/Brooks-Cole, Belmont, CA, USA. ISBN 0-534-42085-0.
- [14] J. Walkenbach. (2003). Excel Charts, Wiley Publishing, Inc.
- [15] D. Arganbright (2005). Enhancing Mathematical Graphical Displays in Excel through Animation, eJSiE 2(1), 120-147.
- [16] W.A.L. Wischniewsky, Movie-like Animation with Excel's Single Step Iteration Exemplified by Lissajous Figures, Spreadsheets in Education (eJSiE), Vol. 3, Iss. 1 [2008], Art. 4.
- [17] S. Abramovich and S. Sugden (2005). Spreadsheet Conditional Formatting: An Untapped Resource for Mathematics Education, *Spreadsheets in Education (eJSiE)*, Vol. 1, Iss. 2 [2007], Art. 1, p.104-124.
- [18] L.R.Wetzel and P.J.Whicker, Quick Correct: A Method to Automatically Evaluate Student Work in MS Excel Spreadsheets, *Spreadsheets in Education (eJSiE)*, Vol. 2, Iss. 3 [2007], Art. 1.
- [19] M. El-Gebeily and B. Yushau, Curve Graphing in MS Excel and Applications, *Spreadsheets in Education (eJSiE)*, Vol. 2, Iss. 2 [2007], Art. 6
- [20] A. Didenko, D. Allison & S. M. Stewart, Using technology to forge the link between formal mathematical definitions and related problem-solving techniques, *Proceeding of the Third Annual Conference for Middle East Teachers of Science, Mathematics and Computing METSMaC*, The Petroleum Institute, Abu Dhabi, UAE, 2007, pp. 119–126.
- [21] A.Didenko, G.Miller, Algebraic and linear algebraic algorithms with EXCEL, *Proceeding of the Third Annual Conference for Middle East Teachers of Science, Mathematics and Computing METSMaC*, The Petroleum Institute, Abu Dhabi, UAE, 2007.
- [22] A. Didenko, D. Allison, Using Microsoft Excel as a Tool for Teaching Calculus to Engineering Students, *Proceedings of the Third International Conference on Mathematical Sciences ICM-2008*, UAEU, Al Ain, UAE, 14pp.
- [23] A. Didenko, D. Allison, G. Miller, Graphing Parametric Curves Using Spreadsheets, *Proceedings of the 34th Conference on Applications of Mathematics in Engineering and Economics AMEE 2008*, Technical University of Sofia, Bulgaria, ISBN 978-0-7354-0598-1, Publisher: American Institute of Physics, pp. 187-195; AIP on-line Conference Proceedings #1067.
- [24] G. Miller, A. Didenko, D. Allison, Orthonormalization by Spreadsheet, *Proceedings of the 6th UAE Mathday*, edited by A.Didenko, ISBN 978-9948-03-668-5, The Petroleum Institute, Abu Dhabi, UAE, 2008, pp. 3
- [25] A. Didenko, D. Allison, G. Miller, The Technology of Drawing a Straight Line, *Proceedings of the 7th UAE Mathday*, Sharjah University, Sharjah, UAE, 2009, p.19
- [26] A. Didenko, D. Allison and G. Miller, Using Spreadsheets as Graphing Tools, *Proceedings of the 6th UAE Mathday*, edited by A.Didenko, ISBN 978-9948-03-668-5, The Petroleum Institute, Abu Dhabi, UAE, 2008.
- [27] A.Didenko, D.Allison, G.Miller., Graphing Polar Equations Using Spreadsheets, *Proceedings of the WSEAS 6th International Conference on Engineering Education*, Rodos, Greece, 2009, pp.108-110.
- [28] A.Didenko, D.Allison, G.Miller., Exploring Taylor Polynomials Using Spreadsheets, *Proceedings of the International Conference on Interactive Computer Aided Learning*, Villach, Austria, 2009, pp.99-102.
- [29] B.S. Gottfried, *Spreadsheet Tools for Engineers Using Excel*, New York: McGraw-Hill Science Engineering, 2005.
- [30] S.C. Bloch, *Excel for Engineers and Scientists, 2e*, New York: John Wiley & Sons, 2003.
- [31] J. Stewart, *Calculus*, California: Brooks/Cole-Thomson, 2003.
- [32] <http://en.wikipedia.org/wiki/Lituus>
- [33] [http://en.wikipedia.org/wiki/Lituus_\(mathematics\)](http://en.wikipedia.org/wiki/Lituus_(mathematics))