Modelling the brazed assembly by the coupling of the eXtended Finite Element and the matched asymptotic development methods

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Abstract: Modelling of the brazed assembly taking account the presence of a brazed joint which is considered as a singularity is proposed. The model is based on the eXtended Finite Element Method (X-FEM) coupled with the matching asymptotic development method (DAR). We consider the behavior of the brazed assembly in two problems separately: mechanical and transient thermal problems. The approach proposed by the DAR method is based on to construct the enriched functions in the X-FEM framework. The fundamental formulation of our coupling is given and illustrated in the 1D case of the brazed assembly. The accuracy of the results obtained by the coupling is evaluated by comparison with the analytical solutions (in the mechanical problem) and with the solutions of the quadrupole method and of the commercial FEM code (ABAQUS) (in the thermal problem).

Key-Words: eXtended Finite Element Method (X-FEM), matched asymptotic development method (DAR), brazed assembly, brazed joint

1 Introduction

In the most recent years, brazing has been used as an efficient and appropriate technique of assemblage, especially in the sector of tooling such as the fabrication of moulds [1]. In these applications, the brazed assembly must withstand high thermomechanical solicitations. So its thermomechanical behavior is a critical point that requires extensive R&D efforts.

In the modelling of the behavior of brazed assemblies taking account the presence of a brazed joint, we meet the difficulties which are usually encountered because of its singularity in the whole assembly. Previously, the brazed joint is either ignored or modelled with an extra fine mesh in the zone around brazed joint [2]. With the latter measure, on the one hand, the mesh must conform to the joint and on the other hand, the computational cost can become enormous. In this situation, we propose the coupling of the eXtended Finite Element Method (X-FEM) and the matching asymptotic development method (DAR) to overcome these difficulties.

The first method X-FEM is based on the idea that the standard finite element approximation space is enriched with specially tailored functions called enriched functions to capture the singularity of a problem. Concretely, within the X-FEM, the elements that incorporate a section of a singularity can be attributed the enriched interpolation functions. At the beginning, the X-FEM has been used for the problems which contain the discontinuities (cracks) with specific enriched functions derived from the fracture mechanics [3]. Then, this method has been applied to the problems of implicit interface [4], holes or inclusions [5].

The second method DAR provides us the approach of solution of the singular problem at two scales: macroscopic (or exterior solution) and microscopic (or interior solution in the "boundary layer") [6]. This method has been firstly applied to the problem of fluid mechanics. After, the DAR has been used in many mechanical sectors, such as: composite materials [7], assemblies of materials [8]…

At the first step, the coupling of two above methods is considered in the modelling of the mechanical behavior of the brazed assembly [9]. This tendency of coupling gives us the promising signals. In the framework of the X-FEM, the enriched functions are chosen to represent as closely as possible the behavior of the singularity. In other words, they should be based on the information which is known *a priori* about the behavior of the domain which contains the singularity. In our idea of the coupling of DAR and X-FEM, this information is exploited from the solutions of the DAR method.

In a second stage, we continue to apply the coupling of X-FEM and DAR to treat the transient thermal behavior of a brazed assemblage.

In this paper, the X-FEM formulation and its coupling with the DAR method are detailed for the one dimensional case of the brazed assembly. In order to evaluate the feasibility and the accuracy of the coupling, we compare the results with those obtained by the analytical methods [10] and by the finite element commercial code ABAQUS.

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2 Problem Formulation

The outline of this paragraph is as follows: in section 2.1, we consider the mechanical problem with the subsections about the approach of DAR method, the formulation of X-FEM and the construction of the enriched functions of X-FEM by using the above solutions of DAR. In section 2.2, let us pass into the thermal problem. We present the formulation of the finite element method (FEM) and then of X-FEM which serve the transient thermal problem. The solutions of DAR and the construction of the enriched functions of X-FEM by coupling with the solution obtained by the DAR in the transient thermal problem are also presented. For the role of reference for the coupling of X-FEM and DAR in the thermal problem, the quadrupole method is mentioned.

2.1 Mechanical problem

2.1.1 Approach of DAR method

The DAR method is used to resolve the problem of singularity. This singularity is characterized by a small parameter ε. For instance, in our model, we introduce the ratio $\varepsilon = e/L$ where e and L denote the thickness of brazed joint and of sheet of base material respectively as the small parameter.

There is a zone around the singularity where the solution of problem is perturbed. We call it the boundary layer.

The DAR method allows us to find the solution of the problem in the form of two asymptotic developments of the small parameter ε (Fig. 1).

The first development, *exterior*, provides us the behavior in the zone "far" from the singularity or valid for the exterior domain:

$$f(x,\varepsilon) = f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \ldots \quad (1)$$

In the above expression, $f_0(x)$ is the solution of the unperturbed problem in which the domain doesn't contain the singularity. The part $(\varepsilon f_1(x) + \varepsilon^2 f_2(x) + \ldots)$ called the perturbation stands for the correction taking into account the singularity and called. When $\varepsilon \to 0$, the perturbation vanishes and we return to the unperturbed solution. In this situation, the small parameter is a determining factor in physics to reduce the considered mathematical model to a simpler model whose solution is an approximation of the solution of the initial model.

The second development, *interior*, is valid "near" the singularity or inside the boundary layer.

$$\overline{f}(x,\varepsilon) = \overline{f_0}(\tilde{x}) + \varepsilon \overline{f_1}(\tilde{x}) + \varepsilon^2 \overline{f_2}(\tilde{x}) + \ldots \quad (2)$$

Here we apply the stretch variable $\tilde{x} = x/\varepsilon$ to consider the behavior in the vicinity of the singularity.

There is a intermediate zone where two developments are matched by the matching condition: the *interior* representation of the *exterior* representation is equal to the *exterior* representation of the *interior* representation.

$$\lim_{x\to 0} (\textit{Outer} \text{ development}) = \lim_{\tilde{x}\to\pm\infty} (\textit{Inner} \text{ development})$$

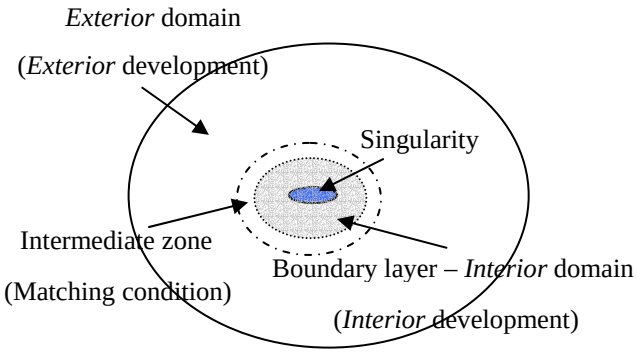

Fig. 1 Two developments of DAR method

The brazed assembly is made of two sheets of base materials 1 and 2 (two sub-domains Ω^1 and Ω^3 respectively) which are assembled by the brazed joint using the filler metal (sub-domain Ω^2). In our study, we only consider the case 1D of this model which is shown in Fig. 2.

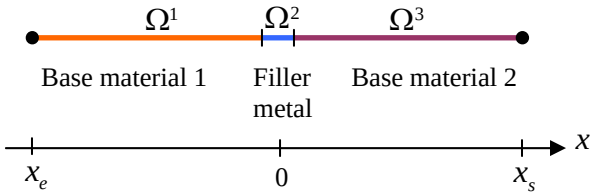

Fig. 2 Model 1D of the brazed assembly

Because of its small thickness, different mechanical and thermal properties compared to those of the base materials, the brazed joint is considered as a singularity.

The exterior field of displacement is given in $\Omega^1(-)$ and $\Omega^3(+)$ respectively in the form of eq. (3):

$$u_{\mp}^{\varepsilon}(x) = u_{\mp}^{0}(x) + \varepsilon u_{\mp}^{1}(x) \quad (3)$$

Where $u_{\mp}^{0}$, $u_{\mp}^{1}$ denote the two first exterior terms of DAR. $u_{\mp}^{0}$ characterize the solution of the unperturbed problem (without Ω^2) while $\varepsilon u_{\mp}^{1}$ stand for the perturbation taking into account the brazed joint.

The brazed joint is zoomed in when applying a change of variable $y = x/\varepsilon$. The zone of the brazed joint is divided into two parts:

$$\begin{cases} \Omega_-^2 : -e/2 \le x \le 0 \Leftrightarrow -L/2 \le y \le 0 \\ \Omega_+^2 : 0 \le x \le e/2 \Leftrightarrow 0 \le y \le L/2 \end{cases} \quad (4)$$

The interior expansion in these two parts is expressed in the form:

$$v_{\mp}^{\varepsilon}(y) = v^0(y) + \varepsilon v_{\Omega_{\mp}^2}^1(y) \quad (5)$$

Where v^0, $v_{\Omega_{\mp}^2}^1$ denote the two first interior terms of the DAR for the two domains $\Omega_-^2(-)$ and $\Omega_+^2(+)$. The two developments are matched with the following condition at their respective limits:

$$\lim_{x\to 0} u^{\varepsilon} = \lim_{y\to\pm\infty} v^{\varepsilon} \quad (6)$$

Each term of the two developments is determined with the simultaneous utilization of the matching condition above and the classical equations of the model (equilibrium equation, constitutive law, continuity condition):

$$u_-^0 = \frac{E_3(u_s - u_e)}{c_2}x + \frac{E_1 x_s u_e - E_3 x_e u_s}{c_2}$$

$$u_+^0 = \frac{E_1(u_s - u_e)}{c_2}x + \frac{E_1 x_s u_e - E_3 x_e u_s}{c_2}$$

$$u_-^1 = \frac{L}{2}\frac{E_3(u_e - u_s)c_1}{E_2 c_2^2}(x - x_e)$$

$$u_+^1 = \frac{L}{2}\frac{E_3(u_e - u_s)c_1}{E_2 c_2^2}(x - x_s)$$

with $c_1 = 2E_1E_3 - E_2E_3 - E_1E_2$; $c_2 = E_1x_s - E_3x_e$

$$v^0 = \frac{E_1 x_s u_e - E_3 x_e u_s}{c_2}$$

$$v_{\Omega_-^2}^1 = \frac{LE_3(u_s - u_e)}{2E_2c_2}\left[\frac{2E_1 y}{L} + (E_1 - E_2) + \frac{x_e c_1}{c_2}\right]$$

$$v_{\Omega_+^2}^1 = \frac{LE_1(u_s - u_e)}{2E_2c_2}\left[\frac{2E_3 y}{L} + (E_2 - E_3) + \frac{x_s c_1}{c_2}\right]$$

where

E_1, E_3 - Young's modulus of base materials

E_2 - Young's modulus of filler metal

x_e, x_s - coordinates of two extremities of domain

u_e, u_s - displacements of two extremities of domain

2.1.2 Formulation of X-FEM

In the modelling of the domain which contains a singularity as the case of brazed assemblage by FEM, we have to ensure the conformity of mesh with the joint. Furthermore, the mesh around the joint must be refined. These create the burdensome cost of calculation.

Thanks to X-FEM, we can resolve these problems. Within the X-FEM framework, an arbitrary mesh can be applied. The field of solution of the domain that contains brazed joint is decomposed into two parts: a standard of finite element approximation (linear) and enriched part to account for the specific singular behavior. We introduce some particular functions called enriched functions to capture the characteristic of the brazed joint. These functions are embedded for the nodes of mesh which are in the vicinity of joint. We call them the enriched nodes.

The discretized approximation of the displacement field is given by:

$$u^h(x) = \underbrace{\sum_{i=1}^{Nn} N_i(x)u_i}_{u_{cla}} + \underbrace{\sum_{k=1}^{Nenr} N_k(x)\psi_k(x)b_k}_{u_{enr}} \quad (7)$$

where Nn is the total number of nodes, $Nenr$ is the number of enriched nodes, N_i, N_k are the shape functions, ψ_k is the enriched function, u_i is the nodal displacement and b_k is the additional degree of freedom. In the formula (7), we distinguish two different parts: u_{cla} is the classical part which presents the behavior of domain without the joint, while u_{enr} is the enriched part that is added to represent the behavior of the joint.

The weak form of boundary value problem consists:

$$\int_{\Omega} \sigma(u) : \varepsilon(v) d\Omega = \int_{\Omega} b.v d\Omega + \int_{\Gamma_t} t.v d\Gamma \quad (8)$$

where σ is the Cauchy's stress, ε is the strain, b is the body force per unit volume, $\bar{u}, t$ are the prescribed displacement and the stress vector on the boundary of domain

By substituting (7) in (8) and applying several changes, we obtain the governing equation:

$$[\mathrm{K}]\{\mathrm{q}\} = \{\mathrm{f}\} \quad (9)$$

where $\{\mathrm{q}\}$ is the vector of generalized nodal displacements, $\{\mathrm{f}\}$ is the vector of exterior force, $[\mathrm{K}]$ is the matrix of rigidity given by eq. (10).

$$[\mathrm{K}] = \sum_{e=1}^{Ne} \int_{\Omega_e}{}^{T} [\mathrm{B}][\mathrm{D}][\mathrm{B}] d\Omega$$
$$\Leftrightarrow K_{ij} = \sum_{e=1}^{Ne} \int_{\Omega_e} E_k \overrightarrow{\nabla N_i} \overrightarrow{\nabla N_j} d\Omega \quad (10)$$

where $[\mathrm{B}]$ is the strain-displacement matrix or the matrix of gradients, $[\mathrm{D}]$ is the Hooke's tensor.

The element approximation of displacement field for element *(IJ)* is given by eq. (11) which can be synthesized in the matrix form eq. (12)

$$u^{(IJ)}(x)=\sum_{i=I,J}N_i(x)u_i+\sum_{i=I,J}N_i(x)\psi_i(x)b_i \tag{11}$$

$$\{\mathrm{u}\}^{(IJ)}=[\mathrm{N}]^{(IJ)}.\{\mathrm{q}\}^{(IJ)} \tag{12}$$

where $[\mathrm{N}]^{(IJ)}$ is the matrix of generalized shape functions.

In our case, we distinguish two types of elements: standard and enriched.

All elements lie in the enriched domain which is defined by a critical radius d_c are enriched. The center of this enriched domain is determined in the middle of brazed joint. The matrix $[\mathrm{N}]^{(IJ)}$ and the vector of generalized nodal displacements $\{\mathrm{q}\}^{(IJ)}$ of enriched element are given by:

$$[\mathrm{N}]^{(IJ)}=\begin{bmatrix}N_I(x) & N_I(x)\psi_I(x) & N_J(x) & N_J(x)\psi_J(x)\end{bmatrix}$$

$$\{\mathrm{q}\}^{(IJ)}=\begin{Bmatrix}u_I & b_I & u_J & b_J\end{Bmatrix}^T \tag{13}$$

The matrix $[\mathrm{N}]^{(IJ)}$ and the vector of generalized nodal displacements $\{\mathrm{q}\}^{(IJ)}$ of standard element are in the habitual form:

$$[\mathrm{N}]^{(IJ)}=\begin{bmatrix}N_I(x) & 0 & N_J(x) & 0\end{bmatrix}$$

$$\{\mathrm{q}\}^{(IJ)}=\begin{Bmatrix}u_I & 0 & u_J & 0\end{Bmatrix}^T \tag{14}$$

2.1.3 Construction of the enriched functions of X-FEM by using the solutions of the DAR method

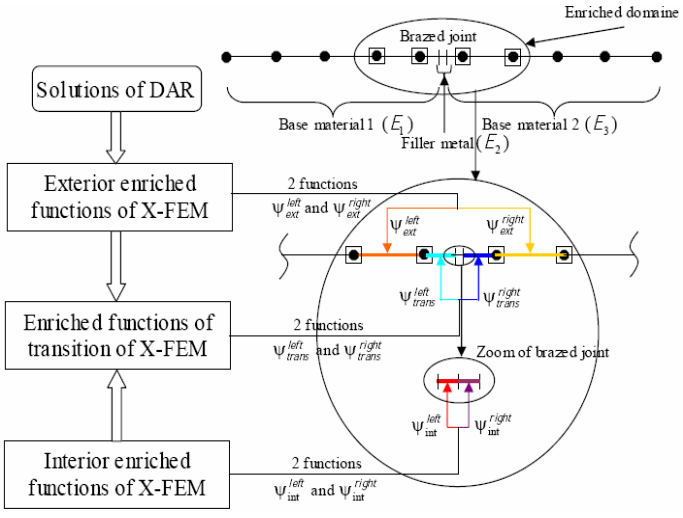

Fig. 3 Idea of coupling X-FEM and DAR

The determination of enriched functions in (7) plays the most essential role within X-FEM. These functions have to represent in the appropriate manner the behavior of whole domain taking into account the presence of the singularity. For this purpose, it is preferable that we know *a priori* the information about the behavior of the domain which contains the singularity. In our work, we are confident of using the solutions of DAR method to acquire this information. The idea of coupling of X-FEM and DAR is presented in Fig. 3.

The enriched domain contains all the enriched nodes around the brazed joint (on the *left* and on the *right*). In this domain, we distinguish three types of enriched functions: the interior ones $\psi_{\mathrm{int}}^{left},\psi_{\mathrm{int}}^{right}$ are valid inside the joint; the exterior ones $\psi_{ext}^{left},\psi_{ext}^{right}$ are used outside the element which contains the joint and the functions of transition $\psi_{trans}^{left},\psi_{trans}^{right}$ are interpolated between the two precedent types.

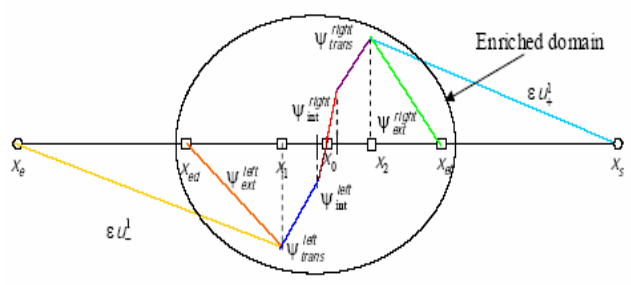

Fig. 4 Relation between the solutions of DAR method and the enriched functions of X-FEM

The exterior enriched functions are calculated from the exterior solutions of DAR and they vanish outside the enriched domain

$$\begin{cases}\psi_{ext}^{left}(x=x_{\mathrm{ed}})=0\\ \psi_{ext}^{left}(x=x_1)=\varepsilon u_-^1(x=x_1)\end{cases}$$

$$\begin{cases}\psi_{ext}^{right}(x=x_{\mathrm{ef}})=0\\ \psi_{ext}^{right}(x=x_2)=\varepsilon u_+^1(x=x_2)\end{cases} \tag{15}$$

The interior enriched functions are determined by using the interior terms of DAR

$$\begin{cases}\psi_{\mathrm{int}}^{left}=\varepsilon v_{\Omega_-^2}^1\\ \psi_{\mathrm{int}}^{right}=\varepsilon v_{\Omega_+^2}^1\end{cases} \tag{16}$$

To link the exterior and interior functions, we introduce the enriched function of transition

$$\begin{cases}\psi_{trans}^{left}(x=x_1)=\psi_{ext}^{left}(x=x_1)\\ \psi_{trans}^{left}(x=x_{\mathrm{ej1}})=\psi_{\mathrm{int}}^{left}(x=x_{\mathrm{ej1}})\end{cases}$$

$$\begin{cases}\psi_{trans}^{right}(x=x_2)=\psi_{ext}^{right}(x=x_2)\\ \psi_{trans}^{right}(x=x_{\mathrm{ej2}})=\psi_{\mathrm{int}}^{right}(x=x_{\mathrm{ej2}})\end{cases} \tag{17}$$

where $x_{\mathrm{ed}},x_{\mathrm{ef}}$ are two nodes of extremity of the enriched domain, x_1,x_2 are two enriched nodes the closest to the joint, $x_{\mathrm{ej1}},x_{\mathrm{ej2}}$ are two ends of the brazed joint.

2.2 Thermal problem

The formulation of FEM and then of X-FEM for the transient thermal problem contains a few remarks which are produced of the dependence of the temperature into the time. So before attacking the application of the coupling of X-FEM and DAR for this problem, we deal its formulations of FEM and of X-FEM.

2.2.1 Formulation of FEM

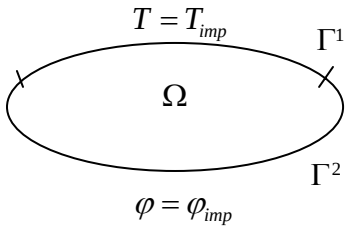

Fig. 5 Domain Ω with the boundary conditions

Considering a habitual case of a body Ω with the boundary conditions in Γ^1 and Γ^2 as in Fig. 5, the strong form of the transient thermal problem writes for the temperature $T(x,t)$ in the form of the equation set (18):

$$\begin{cases} \rho c \dfrac{\partial T}{\partial t} - div(\lambda \overrightarrow{\nabla T}) = \dot{\omega} \text{ dans } \Omega \times]0,t[\\ T(x,0) = T_0 \quad \forall x \in \Omega \\ T = T_{imp} \text{ sur } \Gamma^1 \times]0,t[\\ -\lambda \overrightarrow{\nabla T}.\vec{n} = \varphi_{imp} \text{ sur } \Gamma^2 \times]0,t[\end{cases} \tag{18}$$

where ρ is the mass density, c is the heat specific capacity, $\dot{\omega}$ is the internal source and λ is the thermal conductivity.

The weak form of the problem (18) is given by:

$$\int_\Omega \rho c \frac{\partial T}{\partial t} v d\Omega + \int_\Omega \lambda \overrightarrow{\nabla T}.\overrightarrow{\nabla v} d\Omega = \int_\Omega \dot{\omega} v d\Omega - \int_{\Gamma^2} \varphi_{imp} v d\Gamma \tag{19}$$

To determine the temperature field solution of (19), we partition the domain Ω by a set of Ne sub-elements Ω_e. These sub-elements are connected by Nn nodes. The temperature is approached by the discretized form:

$$T(x,t) = \sum_{i=1}^{Nn} N_i(x) T_i(t) \tag{20}$$

where N_i are shape functions and T_i are the nodal temperatures of the nodes.

By substituting (20) in the weak form (19) and applying a few combinations and eliminating the test functions, it is classical to obtain the matrix form:

$$[\mathrm{C}]\left\{\frac{\partial \mathrm{T}}{\partial t}\right\} + [\mathrm{K}]\{\mathrm{T}\} = \{\mathrm{Q}\} \tag{21}$$

where $\{\mathrm{T}\}$ is the vector of unknowns T_i, $[\mathrm{C}]$ is the matrix of capacity, $[\mathrm{K}]$ is the matrix of conductivity, $\{\mathrm{Q}\}$ is the vector of charge.

The components of these matrix and vectors are calculated as follows (the reader can refer to [11] for more details about the FEM formulation):

$$[\mathrm{C}] = \sum_{e=1}^{Ne} \int_{\Omega_e} \rho c^{\,T} [\mathrm{N}][\mathrm{N}] d\Omega \Leftrightarrow C_{ij} = \sum_{e=1}^{Ne} \int_{\Omega_e} \rho c N_i N_j d\Omega$$

$$[\mathrm{K}] = \sum_{e=1}^{Ne} \int_{\Omega_e} {}^{T}[\mathrm{B}][\mathrm{D}][\mathrm{B}] d\Omega \Leftrightarrow K_{ij} = \sum_{e=1}^{Ne} \int_{\Omega_e} \lambda \overrightarrow{\nabla N_i} \overrightarrow{\nabla N_j} d\Omega$$

$$\{\mathrm{Q}\} = \sum_{e=1}^{Ne} \left(\int_{\Omega_e} \dot{\omega}^{\,T} [\mathrm{N}] d\Omega - \int_{\Gamma^2_e} \varphi_{imp}{}^{\,T} [\mathrm{N}] d\Gamma \right)$$

$$\Leftrightarrow Q_i = \sum_{e=1}^{Ne} \left(\int_{\Omega_e} \dot{\omega} N_i d\Omega - \int_{\Gamma^2_e} \varphi_{imp} N_i d\Gamma \right)$$

To approach the solution of the system (21), we must implement a scheme of temporal derivative. In literature, there are several schemes which give us the approximation in variation of the time. Here, we choose the scheme of two steps time of Crank-Nicholson. The equation (21) is translated into the relation of recurrence:

$$\left(\frac{[\mathrm{C}]}{\Delta t} + \frac{1}{2}[\mathrm{K}] \right) \{\mathrm{T}\}^{m+1} = \left(\frac{[\mathrm{C}]}{\Delta t} - \frac{1}{2}[\mathrm{K}] \right) \{\mathrm{T}\}^{m} + \frac{1}{2}\{\mathrm{Q}\}^{m} + \frac{1}{2}\{\mathrm{Q}\}^{m+1} \tag{22}$$

$\{\mathrm{T}\}^{m+1}$ can be calculated by assuming that $\{\mathrm{T}\}^{m}$ is known.

In the above expression, the step time Δt must satisfy the condition (23) to ensure the stability of the resolution method:

$$\frac{1}{6} \le a \frac{\Delta t}{(\Delta x)^2} \le \frac{1}{2} \tag{23}$$

where a is the thermal diffusivity ($a = \lambda / \rho c$), Δx is the distance between nodes for element near surface with highest temperature gradient.

2.2.2 Formulation of X-FEM

From the expression of FEM in the previous section, we extend to the formulation of the X-FEM for the thermal problem. Let's consider now the domain Ω which contains a singularity, a brazed joint in our study. Because the position of the joint is fixed in the domain,

we accept that it is considered as a singularity within the meaning "spatial" and "no temporal".
Instead of using a mesh which conforms to the brazed joint as in FEM, thanks to X-FEM, we can apply an arbitrary mesh that can overlap the singularity. Moreover, we introduce the particular functions called enriched functions to capture the characteristic of the brazed joint. These functions are embedded to the nodes of mesh which are in the vicinity of joint. We call them the enriched nodes.
The discretized approximation of the temperature field is then given by:

$$T(x,t)=\underbrace{\sum_{i=1}^{Nn} N_i(x)T_i(t)}_{T_{cla}}+\underbrace{\sum_{k=1}^{Nenr} N_k(x)\psi_k(x)b_k(t)}_{T_{enr}} \qquad (24)$$

In the expression (24), the meaning of grandeurs is identical to the eq. (7) except T_i is the nodal temperature.
By substituting (24) in weak form (19), we get the same matrix form as (21) but with different matrix and vectors as in the eq. (25):

$$\left[\mathrm{C}^*\right]\left\{\frac{\partial \mathrm{T}^*}{\partial t}\right\}+\left[\mathrm{K}^*\right]\left\{\mathrm{T}^*\right\}=\left\{\mathrm{Q}^*\right\} \qquad (25)$$

In eq. (25), X^* indeed represents a quantity which is calculated within X-FEM to distinguish this magnitude X in FEM. The formulation of quantities in the equation (23) is given by:

$$\left\{\mathrm{T}^*\right\}=\begin{Bmatrix} T_i \\ b_i \end{Bmatrix}$$

$$\left[\mathrm{C}^*\right]=\sum_{e=1}^{Ne}\int_{\Omega_e}\rho c^{\,T}\left[\mathrm{N}^*\right]\left[\mathrm{N}^*\right]d\Omega$$

$$\left[\mathrm{N}^*\right]=\begin{bmatrix} N_i & N_i\psi_i & N_j & N_j\psi_j \end{bmatrix}$$

$$\left[\mathrm{K}^*\right]=\sum_{e=1}^{Ne}\int_{\Omega_e}{}^{T}\left[\mathrm{B}^*\right]\left[\mathrm{D}\right]\left[\mathrm{B}^*\right]d\Omega$$

$$\left[\mathrm{B}^*\right]=\begin{bmatrix} N_{i,x} & (N_i\psi_i)_{,x} & N_{j,x} & (N_j\psi_j)_{,x} \end{bmatrix}$$

$$\left\{\mathrm{Q}^*\right\}=\sum_{e=1}^{Ne}\left(\int_{\Omega_e}\dot{\omega}^{\,T}\left[\mathrm{N}^*\right]d\Omega-\int_{\Gamma^2{}_e}\varphi_{imp}{}^{T}\left[\mathrm{N}^*\right]d\Gamma\right)$$

In the same way as FEM, to overcome the temporal derivate in (25), we apply Crank-Nicholson's scheme.

$$\left(\frac{\left[\mathrm{C}^*\right]}{\Delta t}+\frac{1}{2}\left[\mathrm{K}^*\right]\right)\left\{\mathrm{T}^*\right\}^{m+1}=\left(\frac{\left[\mathrm{C}^*\right]}{\Delta t}-\frac{1}{2}\left[\mathrm{K}^*\right]\right)\left\{\mathrm{T}^*\right\}^{m}+\frac{1}{2}\left\{\mathrm{Q}^*\right\}^{m}+\frac{1}{2}\left\{\mathrm{Q}^*\right\}^{m+1} \qquad (26)$$

The resolution of (25) becomes a recurrence equation of $\left\{\mathrm{T}^*\right\}^{m+1}$ function of $\left\{\mathrm{T}^*\right\}^{m}$ as in eq. (26).

2.2.3 Coupling of X-FEM and DAR in the 1D thermal problem

2.2.3.1 Approach of DAR method

Let's consider the model of a brazed assembly in the thermal problem (Fig. 6). Because the width of assemblage is very greater than its thickness, we can return to the 1D model of heat transfer problem.

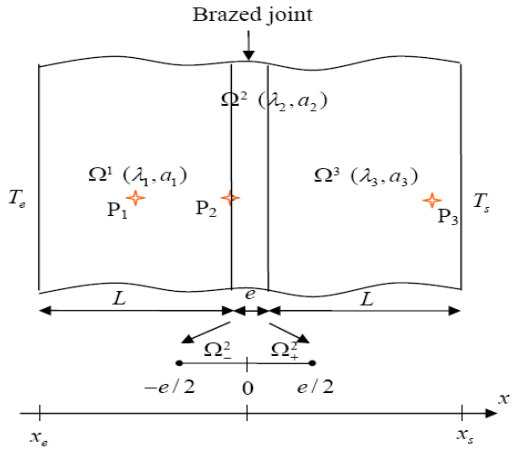

Fig. 6 One-dimensional thermal model of brazed assemblage

By applying the DAR method as we did in the mechanical problem, the temperature is determined at two scales: exterior and interior. The role of temperature field in the thermal problem is identical with that of the displacement field in the mechanical problem.
The exterior temperature fields in $\Omega^1(-)$ and $\Omega^3(+)$ are given respectively by:

$$T_{\mp}^{\varepsilon}(x)=T_{\mp}^{0}(x)+\varepsilon T_{\mp}^{1}(x)+\ldots \qquad (27)$$

where the exterior terms are defined by:

$$\begin{cases} T_{-}^{0}(x)=\dfrac{\lambda_3(T_s-T_e)}{c_4}x+\dfrac{\lambda_1 x_s T_e-\lambda_3 x_e T_s}{c_4} \\ T_{+}^{0}(x)=\dfrac{\lambda_1(T_s-T_e)}{c_4}x+\dfrac{\lambda_1 x_s T_e-\lambda_3 x_e T_s}{c_4} \end{cases}$$

$$\begin{cases} T_{-}^{1}=\dfrac{L}{2}.\dfrac{\lambda_3(T_e-T_s)c_3}{\lambda_2 c_4^2}(x-x_e) \\ T_{+}^{1}=\dfrac{L}{2}.\dfrac{\lambda_1(T_e-T_s)c_3}{\lambda_2 c_4^2}(x-x_s) \end{cases}$$

with $c_3=2\lambda_1\lambda_3-\lambda_2\lambda_3-\lambda_1\lambda_2$; $c_4=\lambda_1 x_s-\lambda_3 x_e$

The interior temperature fields are respectively given in Ω_-^2 and Ω_+^2 by:

$$T_{\mp}^{\varepsilon}(x) = \tau^0(y) + \varepsilon\tau_{\mp}^1(y) + \ldots \tag{28}$$

where the interior terms are defined by:

$$\begin{cases} \tau^0 = \dfrac{\lambda_1 x_s T_e - \lambda_3 x_e T_s}{c_4} \\ \tau^1_{\Omega_-^2} = \dfrac{L\lambda_3(T_s - T_e)}{2\lambda_2 c_4}\left[\dfrac{2\lambda_1 y}{L} + (\lambda_1 - \lambda_2) + \dfrac{x_e c_3}{c_4}\right] \\ \tau^1_{\Omega_+^2} = \dfrac{L\lambda_1(T_s - T_e)}{2\lambda_2 c_4}\left[\dfrac{2\lambda_3 y}{L} + (\lambda_2 - \lambda_3) + \dfrac{x_s c_3}{c_4}\right] \end{cases}$$

2.2.3.2 Construction of the enriched functions of X-FEM by using the solutions of the DAR method

We apply the same strategy of mechanical problem to construct the enriched functions of X-FEM from the solutions of DAR.

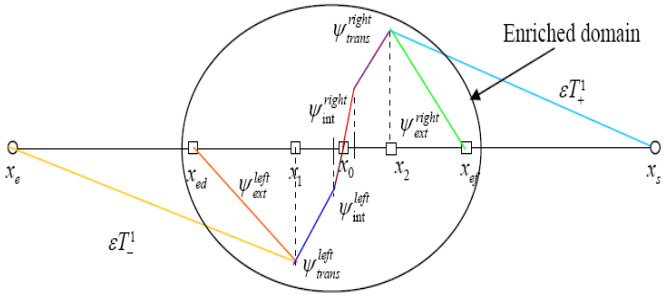

Fig. 7 Determination of the enriched functions of X-FEM for the thermal problem

The determination of these enriched functions from the solutions of DAR method is based on the conditions below:

$$\begin{cases} \psi_{ext}^{left}(x = x_{ed}) = 0 \\ \psi_{ext}^{left}(x = x_1) = \varepsilon T_-^1(x = x_1) \end{cases}$$

$$\begin{cases} \psi_{ext}^{right}(x = x_{ef}) = 0 \\ \psi_{ext}^{right}(x = x_2) = \varepsilon T_+^1(x = x_2) \end{cases}$$

$$\begin{cases} \psi_{int}^{left} = \varepsilon\tau^1_{\Omega_-^2} \\ \psi_{int}^{right} = \varepsilon\tau^1_{\Omega_+^2} \end{cases} \tag{29}$$

$$\begin{cases} \psi_{trans}^{left}(x = x_1) = \psi_{ext}^{left}(x = x_1) \\ \psi_{trans}^{left}(x = x_{ej1}) = \psi_{int}^{left}(x = x_{ej1}) \end{cases}$$

$$\begin{cases} \psi_{trans}^{right}(x = x_2) = \psi_{ext}^{right}(x = x_2) \\ \psi_{trans}^{right}(x = x_{ej2}) = \psi_{int}^{right}(x = x_{ej2}) \end{cases}$$

2.2.4 Quadrupole method

To validate the results we obtained, several methods could have been used such as the conservative averaging method [12] or the method based on the Papoulis Berg method [13] but we used the quadrupole method.

In the thermal research, the quadrupole method developed early on [14, 15] has been successfully used to solve transient thermal problems such as for instance heat transfer in cutting tools [16] or in stratified moulds [17]. In this work, we exploit it to obtain the reference for our coupling of X-FEM and DAR. By the way, we recall some features of this method.

The formulation of quadrupole method provides us with the relation between the input temperature-heat flux vector at the front side and the output vector at the back side through a transfer matrix M :

$$\begin{pmatrix} \theta(x_e, p) \\ \phi(x_e, p) \end{pmatrix} = \underbrace{\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}}_{M} \begin{pmatrix} \theta(x_s, p) \\ \phi(x_s, p) \end{pmatrix} \tag{30}$$

where p is the Laplace variable, $\mathrm{a}, \mathrm{b}, \mathrm{c}$ and d are the components whose forms are interpreted below , θ and ϕ are temperature and heat flux respectively in the Laplace's space.

Fig. 8 below shows the representation of quadrupole method.

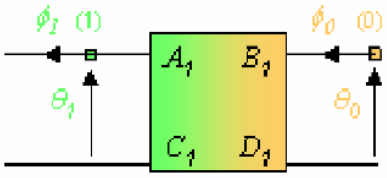

Fig. 8 Representation of quadrupole method

In our case, the brazed assembly contains three layers: plate - joint - plate and the heat flux is orthogonal to the layers. We consider the brazed assembly as the association in cascade of three quadrupoles (Fig. 9).

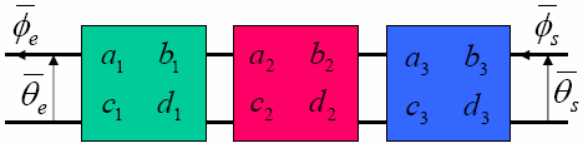

Fig. 9 Association in cascade of three quadrupoles

Then the transfer matrix is the product of three matrices:

$$M = \prod_{i=1}^{3} M_i \tag{31}$$

Each matrix M_i is calculated by:

$$M_i = \begin{pmatrix} \cosh u_i & \sinh u_i / \Delta_1 \\ \Delta_i \sinh u_i & \cosh u_i \end{pmatrix}$$

where $u_i = k_i L$, $\Delta_i = \lambda_i k_i$, $k_i = \sqrt{p/a_i}$

λ_i, a_i are respectively the thermal conductivity and diffusivity of the i^{th} layer.

In eq. (30), two terms are normally determined from the boundary conditions. Then we calculate the remaining unknown. When these two vectors are known, it is easy to deduce the vector of temperature and heat flux at any point in the domain. We have several formulations (Stehfest [18], Hoog [19],…) to return to temporal space to get the functions that depend on time t from the functions that depend on the Laplace variable p.

3 Problem Solution

3.1 Mechanical problem

Let us consider the 1D model of the brazed assembly as depicted in the Fig. 2. Two base plates which are made of steel are brazed together by the nickel - silver alloy as the filler metal. The characteristics of the components of this assemblage are given by:

	Plate	Joint
Thickness (m)	0.04995	10^{-4}
Young's modulus (GPa)	200	74

Tab. 1 Used data for the mechanical problem

The boundary condition at the left extremity of domain is embedded, while at the right, a displacement $U^* = 0.05mm$ is applied.

An analytical calculation and the solution of DAR are chosen as reference for the result of the coupling of X-FEM and DAR (Fig. 10).

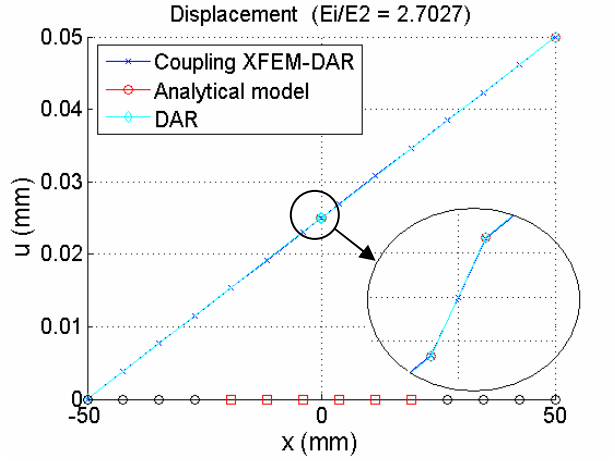

Fig. 10 Solution of displacement of the brazed assembly and its zoom around the joint

The solution of the coupling X-FEM and DAR is very close to that of reference. Through the brazed joint, there is a change of slope of displacement solution which is captured by the enriched functions. The maximal relative error between the coupling of X-FEM and DAR and the analytical model estimates 2.24 x 10^{-5} %.

3.2 Thermal problem

We consider the model of brazed assembly as Fig. 6. The conductivities and the diffusivities summarized in the table 2 below:

	Plate	Joint
Thickness (m)	0.04995	10^{-4}
Conductivity $(W/m.°C)$	15	370
Diffusivity (m^2/s)	3.98 x 10^{-6}	1.061 x 10^{-4}

Tab. 2 Used data for the thermal problem

We assume that the initial temperature is equal to zero for the whole domain:

$$T^0 = T_{ini} = 0°C \qquad (32)$$

The boundary conditions are the Heaviside temperature at the front side and the imposed temperature at the back side:

$$- \; T(x = x_e) = \begin{cases} 0°C & t = 0 \\ 100°C & t > 0 \end{cases}$$

$$- \; T(x = x_s) = 0°C \qquad t \geq 0$$

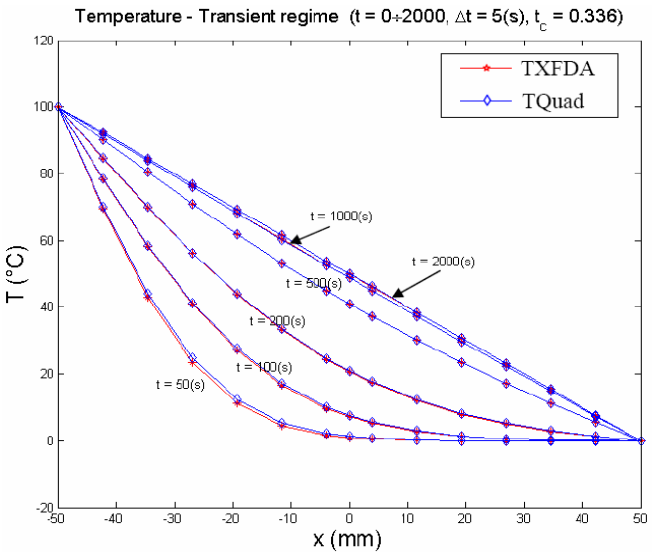

Fig. 11 Profiles of temperature at the time $t = 50, 100, 200, 500, 1000, 2000s$

The temperature distribution is determined in the interval of time $t=[0,2000\,s]$ with the constant step time $\Delta t=5s$. The mesh contains 15 nodes, so $\Delta x=7.69\,(mm)$.

The characteristic time of the problem is calculated by:

$$t_c=\frac{a\Delta t}{\Delta x^2}=\frac{a_1\Delta t}{\Delta x^2}=\frac{3.98\times 5}{7.69^2}=0.336 \qquad (33)$$

This value satisfies the condition of stability (23).

The profiles of temperature at $t=50,100,200,500,1000,2000\,s$ for the whole domain are presented in Fig. 11. '*TXFDA*' and '*TQuad*' denote the temperatures which are calculated by the coupling of X-FEM and DAR and the quadrupole method respectively.

We have chosen several points (P_1, P_2 and P_3) whose positions are indicated in Fig. 6 to represent the evolution of temperature and of heat flux as function of time.

In Fig. 12, we compare the evolution of the temperature as a function of time at the point P_2 which are obtained by two above methods and by the computational code ABAQUS ('*TABA*').

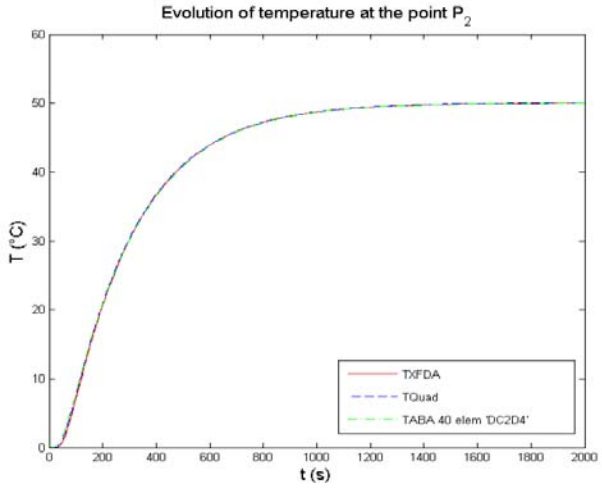

Fig. 12 Evolution of temperature as a function of time at the point P_2

Fig. 13 presents the evolution of the function temperature - time at three points P_1, P_2 and P_3. The symbols '*TXFDA_Pi*' and '*TQuad_Pi*' denote the evolutions of temperature at the point $i\,(i=1,2,3)$ which are obtained by the coupling of X-FEM and DAR and the quadrupole method respectively.

Furthermore, we can see in this figure the error for each couple of curves:

'*Err_Pi*' = '*TXFDA_Pi*' - '*TQuad_Pi*'

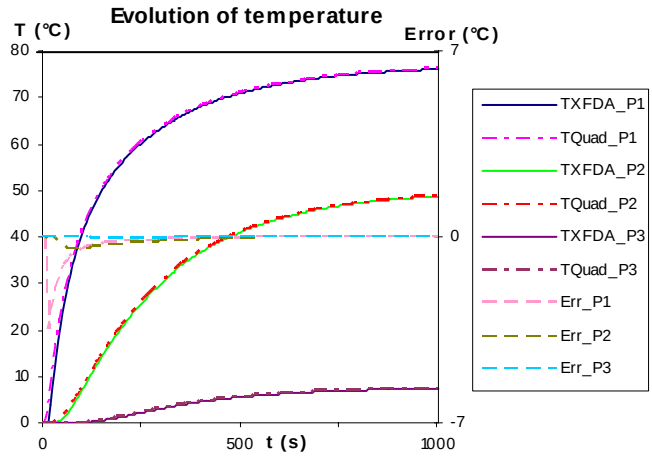

Fig. 13 Evolution of temperature as a function of time at several points and the error between the coupling of X-FEM+DAR and the quadrupole method

In the figure 14, we can see the evolution of the function heat flux density - time at these points (P_1, P_2 and P_3). Identically with the temperature, the labels '*FluXFDA_Pi*' and '*FluQuad_Pi*' denote the evolutions of temperature at the point $i\,(i=1,2,3)$ which are obtained by the coupling of X-FEM and DAR and the quadrupole method respectively. And we have also the error for each couple of curves:

'*Err_Pi*' = '*FluXFDA_Pi*' - '*FluQuad_Pi*'

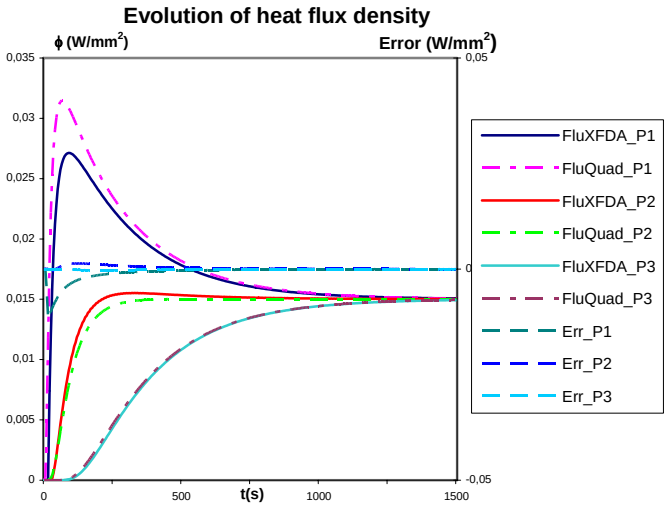

Fig. 14 Evolution of heat flux density as a function of time at several points and the error between the coupling of X-FEM+DAR and the quadrupole method

In this example, we can see that the curves of temperature obtained by the coupling of X-FEM and DAR and by the quadrupole method are identical. Although at several first steps of time, there is a small gap, the results of two methods reach the coherence

rapidly with the increase of time. However, the evolutions of heat flux resolved by two methods show a difference. This can be explained by the accumulation of errors when the heat flux has been derived from the temperature.

4 Conclusion

The coupling of X-FEM and DAR is developed to model the behavior of brazed assembly. We apply it to resolve the mechanical and then the transient thermal problems of the one-dimensional model of brazed assembly. The key idea consists in the inspiration of the solutions of the DAR method in the construction of enriched functions in X-FEM. This trend of coupling between two methods has proved the feasibility and the accuracy.

At the first stage, through the mechanical problem, we construct step by step the formulation of the coupling. The displacement solution obtained by this coupling is very coherent with the analytical solution and the solution of the DAR method itself.

In the second stage, we apply the coupling of X-FEM and DAR for the problem more complicated: the transient thermal problem. And the strategy of coupling always shows its reliability. The profiles of temperature in the whole domain and the evolution of temperature and heat flux density in functions of time at several points of domain which obtained by this coupling are very close to the results of the quadrupole method and of ABAQUS model.

These results encourage us to go on performing a two-dimensional formulation and obtaining finally thermo-mechanical response of the brazed assembly thanks to the coupling of extended finite element and matched asymptotic development methods.

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