A Combined Scheme for Computing Numerical Solutions of a Free Boundary Problem

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Abstract: Numerical schemes for free boundary problems are categorized into two groups: level-set approaches and iterative approaches. In this paper we present a combined approach for computing numerical solutions of a free boundary problem. At first, a rough numerical solution is obtained by a level-set method. Then, using the solution as an initial guess, we use an iterative scheme to obtain more precise solution. To design an iterative scheme, we calculate first variations with respect to boundary perturbation of quantities related to the free boundary problem. Such a variation with respect to domain perturbation is called Hadamard's variation. Since our iterative scheme is designed with Hadamard's variations, it is fast and stable. If the iteration starts with good initial guess obtained by a level-set method, iteration converges almost immediately. Numerical examples show the effectiveness and usefulness of our approach.

Keywords: Filtration Problem, Free boundary problems, Hadamard's variations, Traction method

I. Introduction

Suppose that there are two disjoint water reservoirs separated by a dam made by porous media (earth, for example) (see Figure 1). The different surface levels produces a water flow inside the dam. The problem to find the flow region and the velocity potential function is called the **filtration problem** (**dam problem** or **seepage problem**, etc). In many text books, the filtration problem has been considered as one of the most typical examples of free boundary problems; see [6], [13], [15].

Let us denote the region of the dam by $\mathcal{DAM}$. Numerical schemes for free boundary problems are categorized into two methods; level-set approaches and iterative approaches. In level-set approaches, a free boundary problem is transformed into a problem defined in the whole domain $\mathcal{DAM}$. Typically, the flow region Ω is expressed as $\Omega = \{x \in \mathcal{DAM} | \gamma(x) > 0\}$, where the function γ is a solution of the transformed problem. In iterative approaches, staring from an initial guess

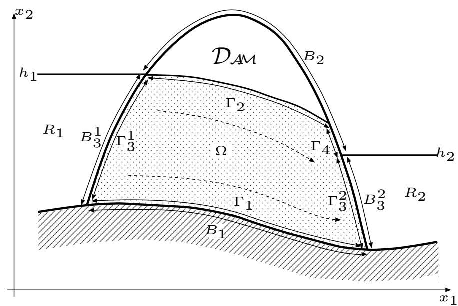

Figure 1: The configuration of the dam.

$\Omega^{(0)}$, the k-th flow region $\Omega^{(k)}$ is updated gradually until $\Omega^{(k)}$ satisfies certain conditions numerically.

The purpose of this paper is to present a combined approach for the filtration problem. At first, we use a level-set method to obtain a numerical solution of the filtration method. Then, using the obtained solution as the initial guess, we adopt an iterative scheme to refine the flow region. For that purpose we use the iterative scheme presented in [17]. Since the iteration starts from a good initial guess and our iterative scheme is defined based on solid mathematics, the algorithm converges quickly to the solution.

The following is the outline of this paper. In Section 2, we give the rigorous definition of the filtration problem. In Section 3 we explain a variational principle of the filtration problem which was introduced in [16]. In the formulation, functionals a, b and $J := a - b$ are defined in a set of subsets of $\mathcal{DAM}$ [1], and is showed that, for an admissible domain Ω, $J(\Omega) = 0$ if and only if Ω is the exact flow region. With this variational principle, we try to update k-th flow region $\Omega^{(k)}$ to $\Omega^{(k+1)}$ so that $J(\Omega^{(k+1)}) < J(\Omega^{(k)})$. To design such an iterative scheme, it is important to know how the functional J would be varied when the free boundary is per-

[1] Elements of this set are called *admissible domains*, which are candidates for the solution of the filtration problem.

turbed. Such variations with respect to boundary perturbation are called **Hadamard's variations**. In Section 4, we give the first Hadamard variations of the functionals a, b and J. In Section 5, using the obtained first variations, we introduce an iterative scheme which is presented in [17]. The iterative scheme is call the **traction method**. The idea of the traction method is given by Azegami [4]. The traction method is the first iterative scheme which uses the first variation of any variational principle of the filtration problem. Since the traction method is based on rigorous mathematical analysis, it is robust and stable. In Section 6, some of level-set approaches are surveyed. In Section 7, a numerical example is given.

II. Definition of the filtration problem

In this section we briefly explain the definition of the two-dimensional filtration (or dam) problem using the notation of [16]. We assume that $\mathcal{DAM}$ is a Lipschitz domain in $\mathbb{R}^2$. We also assume that the boundary $\partial(\mathcal{DAM})$ consists of three parts: B_1, the impervious part; B_2, the part contact with air; and $B_3 = B_3^1 \cup B_3^2$, the part contact with the water reservoirs R_1 and R_2. We assume that the level of the water reservoirs denoted by h_1 and h_2 $(h_1 > h_2)$ are different and there exists a steady water flow inside $\mathcal{DAM}$. We denote Ω as the portion of water in $\mathcal{DAM}$ which not a priori known. The boundary $\partial\Omega$ consists of four parts:

$\Gamma_1 = B_1$	(the impervious part)
$\Gamma_2 \subset \mathcal{DAM}$	(the free boundary)
$\Gamma_3^i = B_3^i$	(the part in contact with water reservoir$R_i,\ i = 1,2)$
$\Gamma_3 = \Gamma_3^1 \cup \Gamma_3^2$	
$\Gamma_4 \subset B_2$	(the part in contact with air)

Let $\Gamma \subset \mathbb{R}^2$ be a curve. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the canonical projection defined by $\pi((x_1, x_2)) := x_1$. In this paper, we say that Γ is a **graph** in the direction of x_2, if $(\pi|_\Gamma)^{-1}(x_1)$ is connected for all $x_1 \in \pi(\Gamma)$. For the configuration of the Lipschitz domain $\mathcal{DAM}$, we assume the following properties in this paper:

(1) There are two reservoirs of water (one of them may be empty) separated by the dam. We assume without loss of generality that the water level of the left-hand side reservoir is higher than that of the other.

(2) Each reservoir contacts the impervious base.

(3) $B_1 \subset \partial(\mathcal{DAM})$ (impervious part) and $B_2 \cup B_3 \subset \partial(\mathcal{DAM})$ (air and water parts) are continuous, piecewise C^2 curves, both are graphs in the direction of x_2, and $B_2 \cup B_3$ lies above B_1.

The problem is to find the flow region Ω and the velocity potential function u of the flow. To define the boundary value of u we introduce the following subsets of $\mathcal{DAM}$. Let ζ_2 be the point where the surface of the left reservoir contacts $\partial(\mathcal{DAM})$. That is, $\zeta_2 = \overline{B_3^1} \cap \overline{B_2}$. Let sufficiently small $\eta > 0$ be taken and fixed. Let us assume that $\mathcal{DAM}$ can be split into two connected subsets by a segment $l \subset \mathcal{DAM}$ such that one of its end points is ζ_2 and the other is on B_1. Suppose that the angle between B_3^1 and l is η. Then, $\mathcal{DAM}^0 \subset \mathcal{DAM}$ is defined as the region between B_3^1 and l (see Figure 2). Set

$$\mathcal{DAM}^1 := \left\{ x = (x_1, x_2) \in \mathcal{DAM} - \mathcal{DAM}^0 \;\middle|\; x_2 \geq h_2 \right\},$$
$$\mathcal{DAM}^2 := \left\{ x = (x_1, x_2) \in \mathcal{DAM} - \mathcal{DAM}^0 \;\middle|\; x_2 < h_2 \right\}.$$

We then define $u^0 \in H^1(\mathcal{DAM})$ by

$$u^0(x) := \begin{cases} h_1 & \text{on } B_3^1, \\ x_2 & \text{in } \mathcal{DAM}^1, \\ h_2 & \text{in } \mathcal{DAM}^2. \end{cases}$$

(In $\mathcal{DAM}^0$, u^0 is defined in an appropriate way.)

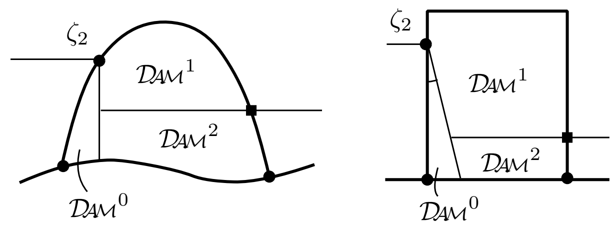

Figure 2: $\mathcal{DAM}^j$ $(j = 0, 1, 2)$.

Then, the **filtration problem** is to find the flow region $\Omega \subset \mathcal{DAM}$ and the piezometric function (velocity potential) u defined on Ω which satisfies the boundary value problem

$$\begin{aligned} &\Delta u = 0 && \text{in } \Omega, \\ &\frac{\partial u}{\partial \boldsymbol{\nu}} = 0 && \text{on } \Gamma_1, \\ &u = u^0 \quad \text{and} \quad \frac{\partial u}{\partial \boldsymbol{\nu}} = 0 && \text{on } \Gamma_2, \\ &u = u^0 && \text{on } \Gamma_3, \\ &u = u^0 \quad \text{and} \quad \frac{\partial u}{\partial \boldsymbol{\nu}} \leq 0 && \text{on } \Gamma_4, \end{aligned} \tag{1}$$

where $\boldsymbol{\nu} := (\nu_1, \nu_2)$ is the unit outer normal vector of $\partial\Omega$. Note that on the free boundary, both Dirichlet's and Neumann's conditions are imposed. In other words, the free boundary is determined so that the both conditions are satisfied at once. The condition $\frac{\partial u}{\partial \boldsymbol{\nu}} \leq 0$ is imposed on Γ_4. The physical meaning of this condition is that water flow comes from inside to outside on Γ_4. This condition is natural and is important for the uniqueness of the exact solution.

In the case that the dam is a rectangle, Baiocchi [5] transformed the problem to a variational inequality. Later, Alt [1] and Brezis-Kinderlehrer-Stampacchia [9] gave different approaches which can treat general situations, and proved the existence of a solution of the filtration problem. The uniqueness of the solution was proved by Alt-Gilardi [3] and Carrillo-Chipot [11]. Note that the above mentioned results are obtained using level-set methods.

III. A Variational Principle of the Filtration Problem

For both mathematical analysis and numerical computation, it would be nice if we have a variational principle of the filtration problem. In this section, we explain a variational principle introduced in [16]. The idea is very simple. Let $\Omega \subset \mathcal{DAM}$ be a candidate of the exact solution of the filtration problem (that is, the true flow region). Let $u_\Omega, w_\Omega \in H^1(\Omega)$ be a two harmonic functions with

$$u_\Omega = u^0, \qquad \frac{\partial w_\Omega}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } \Gamma_2.$$

We suppose that u_Ω, w_Ω satisfy the boundary conditions of (1) on $\Gamma_1 \cup \Gamma_3 \cup \Gamma_4$. If Ω is the exact solution, u_Ω must be equal to w_Ω. If Ω is not the exact solution, the "difference between u_Ω and w_Ω" should represent the distance between Ω and the exact solution in some way. Although, one may take any norm to measure the "difference between u_Ω and w_Ω", we measure the difference in the following manner.

At first, we define the subsets $A(\Omega)$, $B(\Omega) \subset H^1(\Omega)$ by

$$\begin{aligned} A(\Omega) &:= \left\{ v \in \mathcal{K}^*(\Omega) \;\middle|\; v = u^0 \text{ on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \right\}, \\ B(\Omega) &:= \left\{ v \in H^1(\Omega) \;\middle|\; v = u^0 \quad \text{on} \quad \Gamma_3 \cup \Gamma_4 \right\}, \end{aligned}$$

where $\mathcal{K}^*(\Omega)$ is defined by

$$\begin{aligned} \mathcal{K}(\Omega) &:= \big\{ v \in H^1(\Omega) \;\big|\; v = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \\ &\qquad\qquad v \geq 0 \text{ on } \Gamma_4 \big\}, \\ \mathcal{K}^*(\Omega) &:= \big\{ v \in H^1(\Omega) \;\big|\; (\nabla v, \nabla \chi) \leq 0, \\ &\qquad\qquad \forall \chi \in \mathcal{K}(\Omega) \big\}. \end{aligned}$$

Note that for a harmonic function $\chi \in C^{0,1}(\overline{\Omega}) \cap C^2(\Omega)$, χ belongs to $\mathcal{K}^*$ if and only if $\partial\chi/\partial n \leq 0$ on Γ_4 in the sense of distribution. Let a sufficiently large positive number M_0 be taken and fixed. Let also D_Ω denote the Dirichlet integral on Ω:

$$D_\Omega(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 dx.$$

Definition 1 *Under the setting defined so far, a subset $\Omega \subset \mathcal{DAM}$ is called* ***admissible*** *if Ω satisfies the following conditions: (1) Ω is a Lipschitz domain. (2) $\partial\Omega \supset B_1 \cup B_3$. (3) $\partial\Omega - \overline{B_1 \cup B_3}$ is a $C^{0,1}$ curve and is a monotone decreasing graph in the direction x_2. (4) $A(\Omega) \neq \emptyset$ [2] and $\inf_{v \in A(\Omega)} D_\Omega(v) \leq M_0$. We denote by $\mathcal{A_D}$ the set of all admissible domains.*

The functional $a(\Omega)$, $b(\Omega)$, $J(\Omega) : \mathcal{A_D} \to \mathbb{R}$ are defined by

$$\begin{aligned} a(\Omega) &:= \inf_{v \in A(\Omega)} D_\Omega(v), \quad b(\Omega) := \inf_{v \in B(\Omega)} D_\Omega(v), \\ J(\Omega) &:= a(\Omega) - b(\Omega). \end{aligned}$$

[2] If the boundary $\partial\Omega$ is very "wild", $A(\Omega)$ could be empty. So we need to assume $A(\Omega) \neq \emptyset$.

Since $A(\Omega) \subset B(\Omega)$, we have $J(\Omega) \geq 0$.

From the Dirichlet's principle we know that the value $a(\Omega)$ and $b(\Omega)$ are attained by the harmonic functions u_Ω and w_Ω (that is, $a(\Omega) = D_\Omega(u_\Omega)$ and $b(\Omega) = D_\Omega(w_\Omega)$), respectively, which satisfy the boundary conditions

$$\begin{cases} u_\Omega = u^0 \text{ on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \\ \frac{\partial u_\Omega}{\partial \boldsymbol{\nu}} \leq 0 \text{ on } \Gamma_4, \quad \frac{\partial u_\Omega}{\partial \boldsymbol{\nu}} = 0 \text{ on } \Gamma_1, \end{cases} \tag{2}$$

$$w_\Omega = u^0 \text{ on } \Gamma_3 \cup \Gamma_4, \quad \frac{\partial w_\Omega}{\partial \boldsymbol{\nu}} = 0 \text{ on } \Gamma_1 \cup \Gamma_2. \tag{3}$$

We have the following variational principle for the filtration problem:

Theorem 2 *([16, Theorem 2.6]) We have $\inf_{\mathcal{A_D}} J = 0$ for the functional $J : \mathcal{A_D} \to \mathbb{R}$. Moreover, an admissible domain $\Omega \in \mathcal{A_D}$ is a solution of the filtration problem if and only if $J(\Omega) = \inf_{\mathcal{A_D}} J = 0$.*

IV. The Hadamard Variations of $a(\Omega)$ and $b(\Omega)$

By Theorem 2, the filtration problem may be solved (in particular, numerically) by an optimization process. In an optimization process, the boundary would be modified gradually and, therefore, it is very important to know how $a(\Omega)$ and $b(\Omega)$ would vary under perturbation of the domain (or the boundary). Such variations with respect to domain perturbation are called the **Hadamard variations**. In this section we give the first variations of $a(\Omega)$ and $b(\Omega)$ with respect to domain perturbation obtained in [17]. In the next section, we present an iterative scheme using the obtained first variations.

Suppose that we have $\Omega \in \mathcal{A_D}$ and try to modify it. Let a vector field $\mathsf{S} \in W^{1,\infty}(\mathcal{DAM}; \mathbb{R}^2)$ is given. We consider the ordinary equation

$$\begin{aligned} &\frac{dc}{dt}(t) = \mathsf{S}(c(t)), \quad t \geq 0, \\ &c(0) = x, \quad x \in \mathcal{DAM}. \end{aligned}$$

Then, for each $x \in \mathcal{DAM}$ the solution $c(t)$ forms an integral curve. Then, $\mathcal{T}_t(x) := c(t)$ satisfies the following:

- $\mathcal{T}_0(x) = x, \forall x \in \mathcal{DAM}$.
- $\mathcal{T}_t$ is a diffeomorphism of $\mathcal{DAM}$ for sufficiently small $t > 0$.
- $\mathcal{T}_t$ is smooth with respect to t.
- $\mathcal{T}_t$ has the Taylor expansion

$$\mathcal{T}_t(x) = x + t\mathsf{S}(x) + o(t).$$

We use this $\mathcal{T}_t$ as perturbations of $\mathcal{DAM}$.

Now, let $\Omega \in \mathcal{A_D}$ be a candidate of the solution of the filtration problem. Let $u_\Omega \in A(\Omega) \subset H^1(\Omega)$ is the harmonic function which satisfies $a(\Omega) = D_\Omega(u_\Omega)$, that is,

$$\begin{aligned} \Delta u_\Omega &= 0 && \text{in } \Omega, \\ \frac{\partial u_\Omega}{\partial \boldsymbol{\nu}} &= 0 && \text{on } \Gamma_1, \\ u_\Omega &= u^0 && \text{on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \\ \frac{\partial u_\Omega}{\partial \boldsymbol{\nu}} &\le 0 && \text{on } \Gamma_4. \end{aligned} \tag{4}$$

To consider a perturbation of Ω, only the free boundary Γ_2 and Γ_4 would be moved. Hence, we may assume that

$$\operatorname{supp} \mathsf{S} \cap \partial\Omega \subset \Gamma_2 \cup \Gamma_4. \tag{5}$$

The weak form of u_Ω is

$$\begin{aligned} &(\nabla u_\Omega, \nabla v)_\Omega = 0, \quad \forall v \in V_0(\Omega), \\ &u_\Omega = u^0 \quad \text{on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \end{aligned}$$

where $(\cdot,\cdot)_\Omega$ is the inner product of $L^2(\Omega)$ and

$$V_0(\Omega) := \left\{ v \in H^1(\Omega) \;\middle|\; v = 0 \text{ on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \right\}.$$

For a sufficiently small $t > 0$, let $\Omega_t := \mathcal{T}_t(\Omega)$ and suppose that $\Omega_t \in \mathcal{A}_\mathcal{D}$. Set $\Gamma_2^t := \partial\Omega_t \cap \mathcal{DAM}$ and $\Gamma_4^t := \partial\Omega_t \cap B_2$. We also consider the harmonic function u_{Ω_t} such that $a(\Omega_t) := D_{\Omega_t}(u_{\Omega_t})$ where

$$D_{\Omega_t}(u_{\Omega_t}) = \frac{1}{2} \int_{\Omega_t} |\nabla u_{\Omega_t}|^2 \; dx.$$

The harmonic function u_{Ω_t} satisfies the boundary value problem

$$\begin{aligned} \Delta u_{\Omega_t} &= 0 && \text{in } \Omega_t, \\ \frac{\partial u_{\Omega_t}}{\partial \boldsymbol{\nu}} &= 0 && \text{on } \Gamma_1, \\ u_{\Omega_t} &= u^0 && \text{on } \Gamma_2^t \cup \Gamma_3 \cup \Gamma_4^t, \\ \frac{\partial u_{\Omega_t}}{\partial \boldsymbol{\nu}} &\le 0 && \text{on } \Gamma_4^t. \end{aligned}$$

Defining

$$V_0(\Omega_t) := \left\{ v \in H^1(\Omega_t) \;\middle|\; v = 0 \text{ on } \Gamma_2^t \cup \Gamma_3 \cup \Gamma_4^t \right\},$$

the weak form for u_{Ω_t} is

$$\begin{aligned} &(\nabla u_{\Omega_t}, \nabla v)_{\Omega_t} = 0, \quad \forall v \in V_0(\Omega_t), \\ &u_{\Omega_t} = u^0 \quad \text{on } \Gamma_2^t \cup \Gamma_3 \cup \Gamma_4^t. \end{aligned}$$

Note that we have

$$\tilde{v} \in V_0(\Omega_t) \Longleftrightarrow \tilde{v} \circ \mathcal{T}_t \in V_0(\Omega). \tag{6}$$

To show the theorem below, (6) plays an important role. Let $\langle \cdot,\cdot \rangle_{\Gamma_2 \cup \Gamma_4}$ denote the duality pair of $H^{-1/2}(\Gamma_2 \cup \Gamma_4)$ and $H^{1/2}(\Gamma_2 \cup \Gamma_4)$. Then the first variation $\delta a(\Omega)$ can be defined as

$$\delta a(\Omega) := \lim_{t \to +0} \frac{a(\Omega_t) - a(\Omega)}{t},$$

where we can have the following theorem.

Theorem 3 *([17, Theorem 4.1]) Let $\Omega \in \mathcal{A}_\mathcal{D}$ be an admissible domain. Suppose that the perturbation $\mathcal{T}_t(x) = x + t\mathsf{S}(x) + o(t)$ satisfies that $\Omega_t := \mathcal{T}_t(\Omega) \in \mathcal{A}_\mathcal{D}$ for all sufficiently small $t > 0$ and (5). Then, the first variation $\delta a(\Omega)$ is written by*

$$\delta a(\Omega) = \frac{1}{2} \left\langle 1 - \left(\frac{\partial p_\Omega}{\partial \boldsymbol{\nu}} \right)^2, \delta\rho \right\rangle_{\Gamma_2 \cup \Gamma_4},$$

where $p := u_\Omega - x_2$ and $\delta\rho := \mathsf{S} \cdot \boldsymbol{\nu}$ is the normal component of S.

Remark: (1) The function $p_\Omega = u_\Omega - x_2$ represents the water pressure.
(2) If Γ_2 is sufficiently smooth so that $\partial u_\Omega / \partial x_1$ and $\partial u_\Omega / \partial x_2$ exist at almost all points on Γ_2 in the classical sense. Then, the first variation is written as an usual integral over $\Gamma_2 \cup \Gamma_4$:

$$\delta a(\Omega) = \frac{1}{2} \int_{\Gamma_2 \cup \Gamma_4} \left(1 - \left(\frac{\partial p_\Omega}{\partial \boldsymbol{\nu}} \right)^2 \right) \delta\rho ds.$$

Recall that $\Omega \in A_D$ and the harmonic function $w_\Omega \in B(\Omega) \subset H^1(\Omega)$ is a solution of the boundary value problem (3). It satisfies the following boundary problem

$$\begin{aligned} \Delta w_\Omega &= 0 && \text{in } \Omega, \\ w_\Omega &= u^0 && \text{on } \Gamma_3 \cup \Gamma_4, \\ \frac{\partial w_\Omega}{\partial \boldsymbol{\nu}} &= 0 && \text{on } \Gamma_1 \cup \Gamma_2. \end{aligned} \tag{7}$$

Its weak form is

$$\begin{aligned} &(\nabla w_\Omega, \nabla v)_\Omega = 0, \quad \forall v \in V_1(\Omega), \\ &w_\Omega = u^0 \quad \text{on } \Gamma_3 \cup \Gamma_4, \end{aligned}$$

where

$$V_1(\Omega) := \left\{ v \in H^1(\Omega) \;\middle|\; v = 0 \text{ on } \Gamma_3 \cup \Gamma_4 \right\}.$$

We now consider the harmonic function $w_{\Omega_t} \in B(\Omega_t)$ which satisfies the boundary value problem

$$\begin{aligned} \Delta w_{\Omega_t} &= 0 && \text{in } \Omega, \\ w_{\Omega_t} &= u^0 && \text{on } \Gamma_3 \cup \Gamma_4^t, \\ \frac{\partial w_{\Omega_t}}{\partial \boldsymbol{\nu}} &= 0 && \text{on } \Gamma_1 \cup \Gamma_2^t. \end{aligned}$$

The weak form for w_{Ω_t} is

$$\begin{aligned} &(\nabla w_{\Omega_t}, \nabla v)_\Omega = 0, \quad \forall v \in V_1(\Omega_t), \\ &w_{\Omega_t} = u^0 \quad \text{on } \Gamma_3 \cup \Gamma_4^t, \end{aligned}$$

with

$$V_1(\Omega_t) := \left\{ v \in H^1(\Omega_t) \;\middle|\; v = 0 \text{ on } \Gamma_3 \cup \Gamma_4^t \right\}.$$

The difficulty here comes from the fact that

$$\tilde{v} \in V_1(\Omega_t) \Longleftrightarrow \tilde{v} \circ \mathcal{T}_t \in V_1(\Omega) \tag{8}$$

is *not* valid in general since the boundary point $\zeta_5 := \overline{\Gamma}_2 \cap \overline{\Gamma}_4$ may be "peeled off" by the perturbation (see Figure 3).

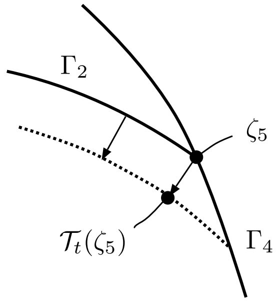

Figure 3: The boundary point ζ_5 may be "peeled off" by the perturbation.

Therefore, we need to impose an additional assumption on perturbation. If (8) holds for all sufficiently small $t \geq 0$, the perturbation $\mathcal{T}_t$ of $\Gamma_2 \cup \Gamma_4$ is said to satisfy the **NPO condition**. (The term "NPO" stands for "Non-Peeling-Off.") Let $b(\Omega_t) := D_{\Omega_t}(w_{\Omega_t})$ where

$$D_{\Omega_t}(w_{\Omega_t}) = \frac{1}{2}\int_{\Omega_t} |\nabla w_{\Omega_t}|^2 \ dx.$$

Then, the first variation $\delta b(\Omega)$ of $b(\Omega)$ is defined by

$$\delta b(\Omega) := \lim_{t \to 0+} \frac{b(\Omega_t) - b(\Omega)}{t}$$

and we have the following theorem.

Theorem 4 *([17, Theorem 5.1]) Let $\Omega \in \mathcal{A}_{\mathcal{D}}$ be an admissible domain and $w_\Omega \in B(\Omega)$ be such that $b(\Omega) = D_\Omega(w_\Omega)$. Suppose that the perturbation $\mathcal{T}_t(x) = x + t\mathsf{S}(x) + o(t)$ satisfies that $\Omega_t := \mathcal{T}_t(\Omega) \in \mathcal{A}_{\mathcal{D}}$ for all sufficiently small $t > 0$ and (5). Moreover, we assume that the NPO condition (8) holds. Then, the first variation $\delta b(\Omega)$ of the functional $b(\Omega)$ is written by*

$$\delta b(\Omega) = \frac{1}{2}\left\langle \left(\frac{\partial w_\Omega}{\partial s}\right)^2, \delta\rho \right\rangle_{\Gamma_2},$$

where $\partial/\partial s$ is tangential derivative along Γ_2 and $\delta\rho := \mathsf{S}\cdot n$ is the normal component of S.

Corollary 5 *([17, Corollary 5.2]) Suppose that all assumptions of Theorem 3 and 4 hold. Then, the first variation $\delta J(\Omega)$ of the functional $J(\Omega) := a(\Omega) - b(\Omega)$ is written by*

$$\begin{aligned}\delta J(\Omega) :=& \lim_{t \to 0+} \frac{J(\Omega_t) - J(\Omega)}{t} \\ =& \frac{1}{2}\left\langle 1 - \left(\frac{\partial p_\Omega}{\partial \boldsymbol{\nu}}\right)^2 - \left(\frac{\partial w_\Omega}{\partial s}\right)^2, \delta\rho \right\rangle_{\Gamma_2}. \qquad (9)\end{aligned}$$

Moreover, $\delta J(\Omega) = 0$ for any sufficiently small $\delta\rho$ if and only if $\Omega \in \mathcal{A}_{\mathcal{D}}$ is the solution of the filtration problem.

V. The Traction Method — An Iterative Scheme

In this section we present an iterative scheme based on the Hadamard variation obtained in the previous section. Suppose that we are trying to obtain the flow region Ω by an iterative scheme. Let $\Omega^{(k)}$ and $\Gamma_2^{(k)} \subset \partial\Omega^{(k)}$ be k-th guess of the flow region and the free boundary, respectively. Since the first variation of the functional $J : \mathcal{A}_{\mathcal{D}} \to \mathbb{R}$ is

$$\begin{aligned}\delta J(\Omega^{(k)}) = \Big\langle 1 - &\left(\frac{\partial p_{\Omega^{(k)}}}{\partial \boldsymbol{\nu}}\right)^2 \\ &- \left(\frac{\partial w_{\Omega^{(k)}}}{\partial s}\right)^2, \delta\rho \Big\rangle_{\Gamma_2^{(k)}},\end{aligned}$$

an intuitive iterative scheme is defined by

$$\begin{aligned}FV(x) &:= 1 - \left(\frac{\partial p_{\Omega^{(k)}}}{\partial n}\right)^2 - \left(\frac{\partial w_{\Omega^{(k)}}}{\partial s}\right)^2, \\ \Gamma_2^{(k+1)} &:= \left\{ x + \epsilon FV(x)\boldsymbol{\nu}(x) \;\middle|\; x \in \Gamma_2^{(k)} \right\}, \qquad (10)\end{aligned}$$

for $x \in \Gamma_2^{(k)}$, where ϵ is a positive dumping parameter and $\boldsymbol{\nu}(x)$ is the unit outer normal vector at $x \in \Gamma_2^{(k)}$. This scheme (10) might be called a *steepest descent method*. However, numerical experiments show that this scheme does not work at all even when ϵ is set very small. After a several iterations, $\Gamma_2^{(k)}$ becomes very "jagged" and computation cannot be carried out any more.

We next propose another iterative scheme which is defined in the following way. Let $z^{(k)} \in H^1(\Omega^{(k)})$ be the solution of the boundary value problem:

$$\begin{aligned} \Delta z^{(k)} &= 0 && \text{in } \Omega^{(k)}, \\ z^{(k)} &= 0 && \text{on } \Gamma_3 \cup \Gamma_4^{(k)}, \\ \frac{\partial z^{(k)}}{\partial \boldsymbol{\nu}} &= 0 && \text{on } \Gamma_1, \\ \frac{\partial z^{(k)}}{\partial \boldsymbol{\nu}} &= FV && \text{on } \Gamma_2^{(k)}. \end{aligned} \qquad (11)$$

Then, the iteration is defined by

$$\Gamma_2^{(k+1)} := \left\{ x - z^{(k)}(x)\boldsymbol{\nu}(x) \;\middle|\; x \in \Gamma_2^{(k)} \right\}.$$

The method is called the **traction method** and was presented by Azegami (see [4] [14] and the references therein) as a numerical iterative scheme for optimal shape design. Numerical experiments show that the traction method works very well for the filtration problem. Beginning from a suitably defined initial guess, the iteration converges smoothly to a numerical solution.

In the following, we point out the two significant natures of the traction method. Firstly, the traction method decreases the value of $J(\Omega)$ in its iterative process. Let $\Omega \subset \mathcal{DAM}$ be an admissible domain. Suppose that the perturbed domain Ω_τ is defined by the traction method

$$\Gamma_2^\tau := \left\{ x - \tau z(x)\boldsymbol{\nu}(x) \;\middle|\; x \in \Gamma_2 \right\}, \quad \tau > 0,$$

where $z(x)$ is a solution of the boundary value problem similar to (11). Letting $\delta\rho := -z$ and $FV := \partial z/\partial n$ on Γ_2 and $\delta\rho := FV := 0$ elsewhere on $\partial\Omega$, we have

$$\begin{aligned}\langle FV, \delta\rho\rangle_{\Gamma_2} &= \left\langle \frac{\partial z}{\partial n}, (-z) \right\rangle_{\Gamma_2} \\ &= \left\langle \frac{\partial z}{\partial n}, (-z) \right\rangle_{\Omega} = -\int_\Omega |\nabla z|^2 dx\end{aligned}$$

and

$$\begin{aligned}J(\Omega_\tau) =& J(\Omega) + \tau\, \delta J(\Omega) + o(\tau) \\ =& J(\Omega) + \tau \langle FV, \delta\rho\rangle_{\Gamma_2} + o(\tau) \\ =& J(\Omega) - \tau \int_\Omega |\nabla z|^2 dx + o(\tau).\end{aligned}$$

Therefore, we may expect

$$J(\Omega_\tau) < J(\Omega)$$

at each step of the traction method. This nature of the traction method is already pointed out by Kaizu and Azegami [14] in a different context.

Secondly, numerical experiments suggest that the traction method seems to have a stabilizing and smoothing effect of the free boundary. Although, the mechanism of this effect of the traction method is not understood completely at this point, we here give a partial explanation. Let the k-th guess $\Gamma_2^{(k)}$ of the free boundary is in $C^{1,\alpha}$ class ($0 < \alpha < 1$). Then, it follows from the regularity theory of linear elliptic PDEs that

$$\begin{aligned}&u_{\Omega^{(k)}}, p_{\Omega^{(k)}}, w_{\Omega^{(k)}} \in C^{1,\alpha}(\Omega^{(k)} \cup \Gamma_2^{(k)}) \quad \text{and} \\ &FV := 1 - \left(\frac{\partial p_{\Omega^{(k)}}}{\partial \boldsymbol{\nu}}\right)^2 - \left(\frac{\partial w_{\Omega^{(k)}}}{\partial s}\right)^2 \in C^{0,\alpha}(\Gamma_2^{(k)}).\end{aligned}$$

Since the Neumann-Dirichlet map

$$C^{0,\alpha}(\Gamma_2) \ni FV = \frac{\partial z^{(k)}}{\partial n} \mapsto z^{(k)} \in C^{1,\alpha}(\Gamma_2)$$

is used in the iterative procedure of the traction method, the updated $\Gamma_2^{(k+1)}$ is in $C^{1,\alpha}$ class again. Hence, the traction method at least preserves the smoothness of the free boundary. If $\Gamma_2^{(k)}$ is updated by the "steepest descend method" (10), however, $\Gamma_2^{(k+1)}$ is only in $C^{0,\alpha}$. Probably, this is a reason of the unstable behaviour of the scheme (10).

VI. Level-Set Approaches

In this section, we survey level-set approaches briefly. There are two types of level-set approaches; one of them reformulates the problem into a variational inequality when the dam $\mathcal{DAM}$ is a rectangle. In the other level-set approach, the filtration problem is reformulated into a problem to find a pair (p, γ), where p is the pressure of the flow and γ is the characteristic function of the flow region. In this section we explain them briefly.

In the case that the dam is a rectangle, Baiocchi [5] transformed the problem into a variational inequality, and show the existence and uniqueness of the solution of the filtration problem (see also [15]). Let $a > 0$, $h_1 > h_2 > 0$ be positive constants. Let $\mathcal{DAM} := (0, a) \times (0, h_1)$ be a rectangle dam. Let the function $g \in H^{2,\infty}(\mathcal{DAM})$ be defined by

$$g(x_1, x_2) := \begin{cases} \frac{a-x_1}{2a}(h_1 - x_2)^2 + \frac{x_1}{2a}(h_2 - x_2)^2, \\ \qquad\qquad\qquad\qquad 0 < x_2 < h_2 \\ \frac{a-x_1}{2a}(h_1 - x_2)^2, \quad h_2 < x_2 < h_1 \end{cases}$$

Let $K \subset H^1(\mathcal{DAM})$ be defined by

$$\begin{aligned}K := \{v \in H^1(\mathcal{DAM}) :& v \geq 0 \text{ in } \mathcal{DAM}, \\ & v = g \text{ on } \partial(\mathcal{DAM})\}\end{aligned}$$

From the theory of variational inequalities (see [15]), there exists a unique solution $w \in K$ of the variational inequality

$$\int_{\mathcal{DAM}} \nabla w \cdot \nabla(v - w) \geq -\int_{\mathcal{DAM}} (v - w),\ \forall v \in K.$$

Then, it is shown that the domain $\Omega := \{x \in \mathcal{DAM} : w > 0\}$ is the desired flow region and $u := x_2 - \partial w/\partial x_2$ is the desired velocity potential. Unfortunately, this beautiful theory works only for the cases that the both sides walls of the dam is vertical.

Later, Alt [1] and Brezis-Kinderlehrer-Stampacchia [9] gave a different approach which can treat general situations, and proved the existence of a solution of the filtration problem. The uniqueness of the solution was proved by Alt-Gilardi [3] and Carrillo-Chipot [11]. In this approach, the filtration problem is formulated to find a pair (p, γ) as is stated in the following. Define the function p^0 by

$$p^0 := \begin{cases} h_i - x_2 & \text{on } B_3^i, \quad i = 1, 2 \\ 0 & \text{on } B_2. \end{cases}$$

Let $e := (0, 1)$.

Problem 6 *Find a pair (p, γ), where $p \in H^1(\mathcal{DAM})$, $\gamma \in L^\infty(\mathcal{DAM})$ such that*

$$0 \leq \gamma \leq 1, \quad \gamma = 1 \textit{ on } \{p \geq 0\}$$

$p = p^0$ on $B_2 \cup B_3$ such that

$$\int_{\mathcal{DAM}} \nabla\zeta \cdot (\nabla p + \gamma e) \leq 0, \quad \forall \zeta \in H^1(\mathcal{DAM})$$

with $\zeta \geq 0$ on B_2 and $\zeta = 0$ on B_3.

As stated above, it has been proved that there exists a unique solution (p, γ). Also, it is shown that $\{p > 0\}$ is the desired flow region, γ is the characteristic function of the region $\{p > 0\}$: $\gamma = \chi_{\{p>0\}}$, and $u := p + x_2$ is the velocity potential of flow inside $\mathcal{DAM}$ (see [13] for detail).

As long as the authors know, any level-set approaches for the filtration problem so far are modifications of one of the above mentioned theorems. Numerical analysis for the filtration problem as a variational inequality was started by Baiocchi and his school. For their works and the development during 1970s and 80s on this subject, see, for example, [7] and [6] and references therein. The first numerical scheme based on the formulation stated as Problem 6 is given by Alt himself [2]. For other numerical schemes based on Problem 6, see, for example, [10], [8]. It seems that numerical schemes based on the formulation of Problem 6 have not been considered thoroughly. Further researches, therefore, are desired and expected.

VII. A Numerical Example

In this section, we give a numerical example which show the effectiveness of our combined approach. Let positive numbers $h_1 > h_2 > 0$, $a > 0$ be given. As $\mathcal{DAM}$ we take a rectangle $\mathcal{DAM} := (0, h_1) \times (0, a)$ (see Figure 4).

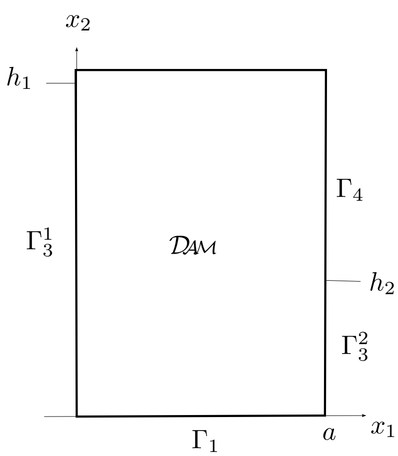

Figure 4: A rectangle dam.

As is stated in the previous section, the filtration problem is reformulated as a variational inequality if $\mathcal{DAM}$ is rectangle. The existence and the uniqueness, therefore, are proved nicely. First, we compute the solution of the variational inequality and use it to obtain the initial guess for iterative scheme. We set the values $a = 1.62$, $h_1 = 3.22$, $h_2 = 0.84$. In Figure 5 we show the numerical solution of the variational inequality. Suppose that $w \in H^1(\mathcal{DAM})$ is the solution of the variational inequality. Then the flow region Ω is represented as $\Omega = \{x \in \mathcal{DAM} : w(x) > 0\}$. In Figure 5, therefore, we draw all triangle elements on which the finite element solution is positive. The union of such elements can be regarded as a numerical approximation of the flow region.

Then, we use the approximated region as an initial region for the traction method (Figure 6).

After several steps, the traction method converges smoothly, and we obtain a numerical solution.

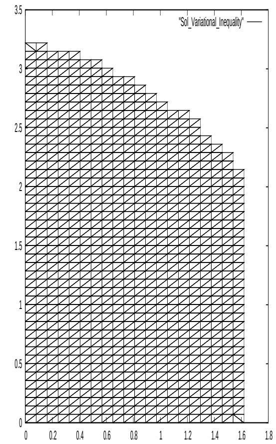

Figure 5: The numerical solution of the variational inequality.

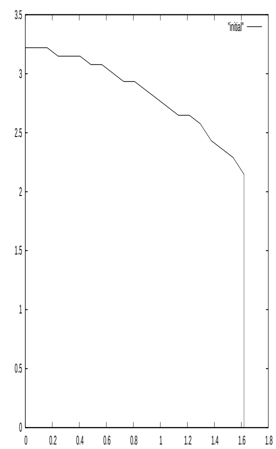

Figure 6: The initial guess made by the variational inequality.

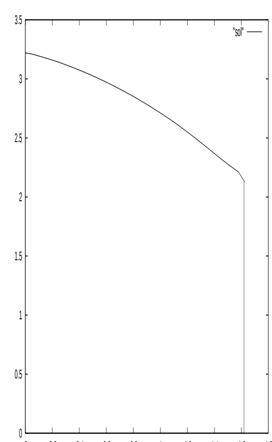

Figure 7: The numerical solution of the traction method.

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