

# Mathematical Model of Control Valves of Three-Tank System

Petr Chalupa, Jakub Novák, Vladimír Bobál

**Abstract**—The paper is focused on development of a mathematical model of valves of a hydraulic system. Laboratory three tank model (Amira DTS200) was investigated and characteristics of its valves were measured. Consequently, the process of creating a mathematical model of the valves is described in detail. The three tank system is a classical modeling task but this paper focuses on nonlinearities which are present in real system and other differences between ideal mathematical model and real-time system. Especially hysteresis represents a big difference between ideal mathematical model and real-time system. Even though all valves the system is equipped with are of the same type, big differences were observed between their characteristics. The approach to modeling of the system is not restricted to the particular system but can be used for many real-time hydraulic systems.

**Keywords**—Control valve, Gray box, Hysteresis, Identification, Modeling.

## I. INTRODUCTION

**A**LMOST all current control algorithms are based on a model of a controlled plant [1]. Some information about controlled plant is necessary for design of a controller with satisfactory performance. A plant model can be also used to investigate properties and behavior of the modeled plant without a risk of damage of violating technological constraints of the real plant. There are two basic approaches of obtaining plant model: the black box approach and the first principles modeling.

The black box approach [2], [3] is based on analysis of input and output signals of the plant. Usage the same identification algorithm for wide set of different controlled plants is the main advantages of this approach. The knowledge of physical principle of controlled plant and solution of set of mathematical equation is not required. Main drawback of a black box model persists in fact that it is generally valid only for signals it was calculated from.

The first principle modeling provides general models valid for wider range of plant inputs and states. The model is created by analyzing the modeled plant and combining

physical laws [4]. But there are usually a lot of unknown constants and relations when performing analysis of a plant.

The paper uses combination of both methods. Basic relations are derived using mathematical physical analysis. Values of model parameters are identified on the basis of real-time measurements. The goal of the work was to obtain a mathematical model of the valves of DTS200 Three-Tank System [5] and to design the models in MATLAB-Simulink environment. The DTS200 laboratory equipment was developed by Amira GmbH, Duisburg, Germany and serves as a real-time model of different industrial systems concerning liquid transport.

The models of valves serve as a part in process of creating a model of whole DTS200 system. The major reason for creating the model of this laboratory equipment are big time constants of the plant and thus time consuming experiments. A model, which represents the plant well, can considerably reduce testing time of different control approaches. Then only promising control strategies are applied to the real plant and verified.

A tank system with valves occurs often in industrial practice and was investigated by many researchers [6], [7]. Flow of liquid through pipes is studied in [8].

The paper is organized as follows. Section 2 presents the modeled system – Amira DTS200. Derivation of initial ideal using first principles modeling is carried out in Section 3. Section 4 and 5 presents characteristics and calibration water level sensors and pumps respectively. Section 6 consists of results of measurements of valves.

## II. THE DTS200 SYSTEM

The photo of main part of Amira DTS200 system is shown in Fig. 1.



Fig. 1. Amira DTS200 – three tank system

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The system consists of three interconnected cylindrical tanks, two pumps, six valves, pipes, water reservoir in the bottom, measurement of liquid levels and other elements. Both pumps pump water from the bottom reservoir to the top of the left and right tanks. Valve positions are controlled and measured by electrical signals, which allow precision setting of their position.

A simplified scheme of the system is shown in Fig. 2. The pump P1 controls the inflow to tank T1 while the pump P2 controls the liquid inflow to tank T2. There is no pump connected to the middle tank Ts. The characteristic of the flow between tank T1 and tank Ts can be affected by valve V1, flow between tanks Ts and T2 can be affected by the valve V2 and the outflow of the tank T2 can be affected by valve V3. The system also provides the capability of simulating leakage from individual tanks by opening the valves V4, V5 and V6.

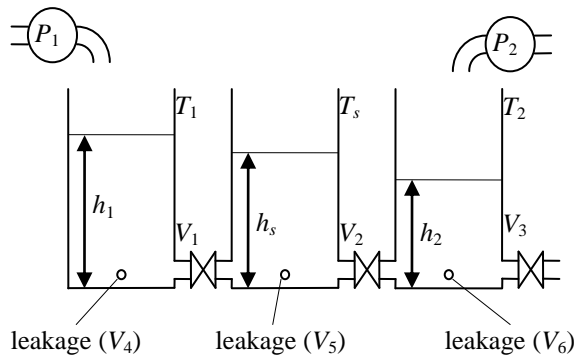


Fig. 2. Scheme of three tank system Amira DTS200

Pumps are controlled by analogue signal in range from -10V to 10V. Heights of water level are measured by pressure sensors. Each valve is operated by two digital signals which control motor of particular valve. First signal orders to start closing of the valve while the second signal is used for opening of the valve. If none of the signals is activated the valve remains in its current position. Each valve also provides three output signals: analogue voltage signal correspond to the current position of the valve and two informative logical signals which states that the valve is fully opened or fully closed respectively.

The overall number of inputs to the modeled plant DTS200 is 14:

- 2 analogues signals controlling the pumps,
- 12 digital signals (2 for each of the 6 valves) for opening / closing of the valves.

The plant provides 21 measurable outputs which can be used as a control feedback or for measurements of plant characteristics:

- 3 analogue signals representing level heights in the three tanks,
- 6 analogues signals representing position of the valves,
- 12 logical signals (2 for each of the 6 valves) stating that corresponding valve is fully opened / closed.

### III. INITIAL IDEAL MODEL

This chapter is focused to derivation of mathematical model of a valve. This derivation is based on ideal properties of individual components.

The ideal flow of a liquid through a pipe can be derived from Bernoulli and continuity equations for ideal liquid:

$$\left. \begin{aligned} \Delta h g &= \frac{1}{2} v^2 \\ q &= S_v \sqrt{2g} \sqrt{\Delta h} \\ q &= S_v v \end{aligned} \right\} \quad (1)$$

where  $\Delta h$  is a difference between liquid levels on both sides of the pipe (e.g. difference between levels of tanks that are interconnected by the pipe),  $g$  is the standard gravity,  $v$  is the liquid velocity and  $S_v$  is the flow space of the pipe. The flow space  $S_v$  is controlled by the valve position  $p$ .

$$S_v = p \cdot S_{v_{\max}} \quad 0 \leq p \leq 1 \quad (2)$$

where  $S_{v_{\max}}$  is the maximal flow area of the valve.

Since the flow through a valve depends only on the level difference, the valve position and constants representing pipes and cylindrical tanks, the change of water level in tank  $T_1$  can be written as follows:

$$\frac{dh_1}{dt} = k_1 \sqrt{|h_1 - h_s|} \cdot \text{sign}(h_1 - h_s) - k_4 \sqrt{h_1} \quad (3)$$

The area of all three tanks is the same and is symbolized by  $S_T$ . The  $k$  is a parameter representing valve position

$$k_i = p_i \frac{S_{v_{\max}} \sqrt{2g}}{S_T} \quad i = 1, 2, \dots, 6 \quad (4)$$

and  $q$  represents inflow as change of water level in time:

$$q_i = \frac{q'_i}{S_T} \quad i = 1, 2 \quad (5)$$

Similar equations can be derived for the other two tanks. The model obtained by using ideal properties and behaviour of plant parts if further referred as "ideal model". This model of whole three tank system is successfully used in many control system studies as a demonstration example [9], [10], [11].

### IV. CHARACTERISTICS OF THE VALVES

As stated in Section 2, each of plant's 6 valves is driven by two dedicated logical signals. These signals are used for starting valve's motor in closing or opening direction respectively. If none signal is activated the valve remains in its current position. Activation of both signals at in a particular time represents an invalid state and valve motor is stopped.

Each valve provides three output signals. The current valve position is determined by analogue signal. Higher values of signal represent closed valve and lower values represent opened valve. The other two signals are logical and state that valve is opened or closed respectively.

A. Valve limits and speed

Process of opening all valves at once from fully closed state to fully opened state is presented in Fig. 3.

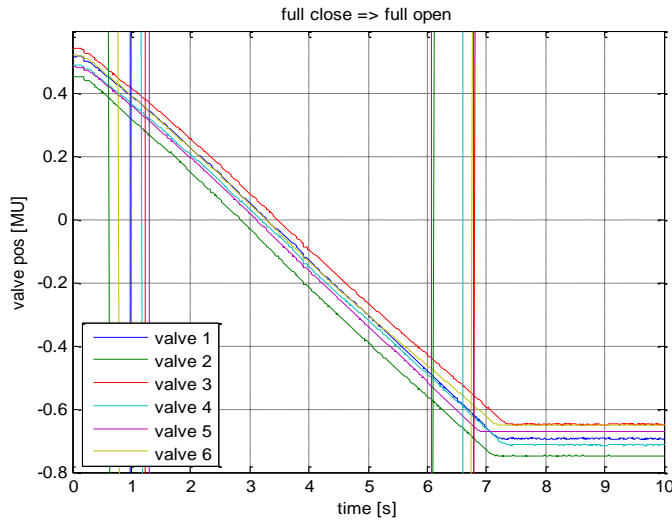


Fig. 3. Closing all valves in full range

This process represents moving of valve position in full range of its hard constraints. The vertical lines in left part of Fig. 3 represent changes the “opened” signals of individual valves. Before these signals drop down the valves are said to be opened. The vertical lines in the right part of Fig. 3 represent the changes of “closed”. From these lines onward, the valves are said to be closed.

It can be observed that the initial and final positions of the valve as well as the positions corresponding to changes of “opened” and “closed” signal differ. But all the valves are moving at almost the same speed  $v_{valve}$ .

$$v_{valve} = -0.175 \text{ MU/s} \tag{6}$$

Valve positions corresponding to hard constraints and validity of “opened” and “closed” state are summarized in Table I.

Table I  
Valve positions for important states

Valve no.	full closed [MU]	“closed” signal change [MU]	“opened” signal change [MU]	full open [MU]
1	0.5199	0.4066	-0.6065	-0.6945
2	0.4550	0.3880	-0.5719	-0.7493
3	0.5462	0.3996	-0.5423	-0.6501
4	0.4926	0.3594	-0.5751	-0.7151
5	0.4868	0.3296	-0.5157	-0.6718
6	0.5242	0.4416	-0.5698	-0.6525

B. Valve flow parameter for outflow valves

Valve flow parameters  $k_i$  as appear in (3) were computed from measurements of draining through individual valves which are connected to outflow pipes ( $V_3, V_4, V_5$  and  $V_6$ ). The

draining of a tank to the reservoir situated below the tanks is described by differential equation based on (3):

$$\frac{dh(t)}{dt} = -k\sqrt{h(t)} \tag{7}$$

Integrating in an appropriate time range leads to the equation of time course of water level:

$$h(t) = \frac{k^2}{4} \cdot t^2 - k\sqrt{h(0)} \cdot t + h(0) \tag{8}$$

where  $h(0)$  is initial water level. An example of draining is presented in Fig. 4. At the beginning of the experiment, the tank was full and all valves were closed, then valve  $V_4$  was partially opened, its position was recorded and time course of water level height was measured.

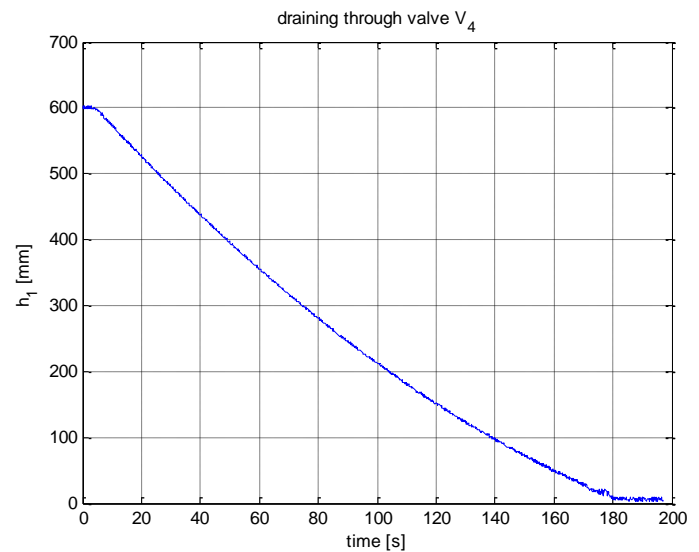


Fig. 4. Draining of tank  $T_1$  through valve  $V_4$

It is obvious that parabola depicted in Fig. 4 would continue below zero contrary to (8). A term corresponding to the vertical length of outflow pipe  $h_0$  was added to the model. The vertical length  $h_0$  is depicted in Fig. 5.

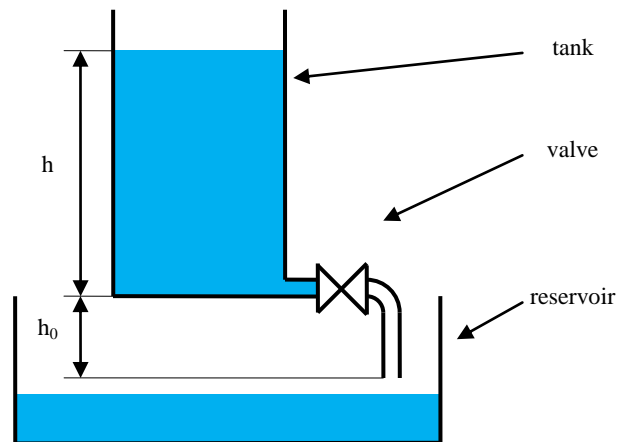


Fig. 5. Vertical length of outflow pipe ( $h_0$ )

Due to mechanical configuration of the plant, the value of  $h_0$  for outflow valves  $V_3, V_4, V_5,$  and  $V_6$  cannot be measured directly. But it can be identified from draining course (Fig. 4). To encapsulate  $h_0$  into model, equations (7) and (8) were superseded:

$$\frac{dh(t)}{dt} = -k\sqrt{h(t)+h_0} \tag{9}$$

$$h(t) = \frac{k^2}{4} \cdot t^2 - k\sqrt{h(0)+h_0} \cdot t + h(0)$$

A second order polynomial (parabola) was fitted to an appropriate interval of draining data in least mean squares sense. The MATLAB function *polyfit* was used for this task. Parabola fitting is presented in Fig. 6.

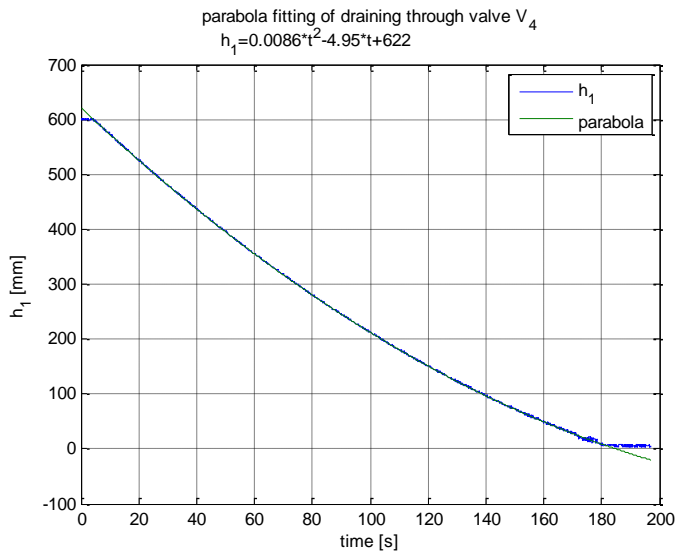


Fig. 6. Parabola fitting to the draining course

Values of  $k$  and  $h_0$  can be easily obtained from polynomial coefficient according to (9). Valve can be closed to different positions at the beginning of draining experiment and relation between valve position and value of  $k$  can be achieved.

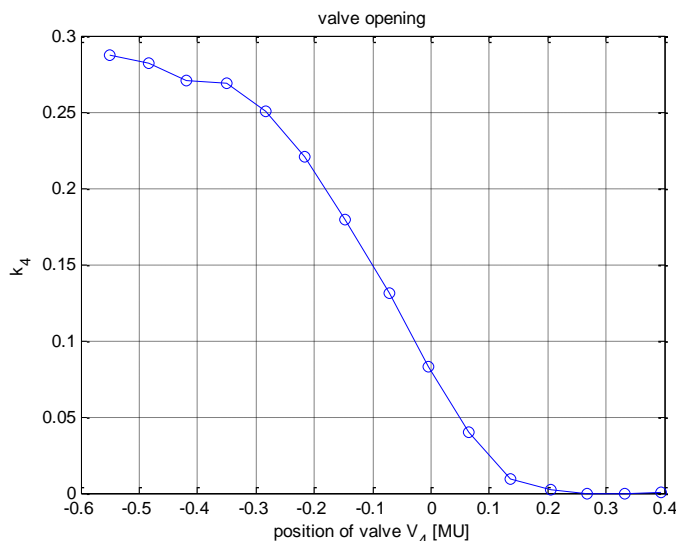


Fig. 7. Relation between valve position and  $k_4$

This relation for one set of experiment on valve  $V_4$  is presented in Fig. 7 where circles represent individual experiments. The characteristic is not strictly linear. It contains saturation of fully closed and fully opened valve. Transitions to saturation states are smooth.

Experiments with the same configuration were performed for all outflow valves. The results obtained from measurement of valve  $V_6$  are presented in Fig. 8. The upper graph depicts the course of  $k_6$  and values of corresponding  $h_0$  are shown in the lower graph.

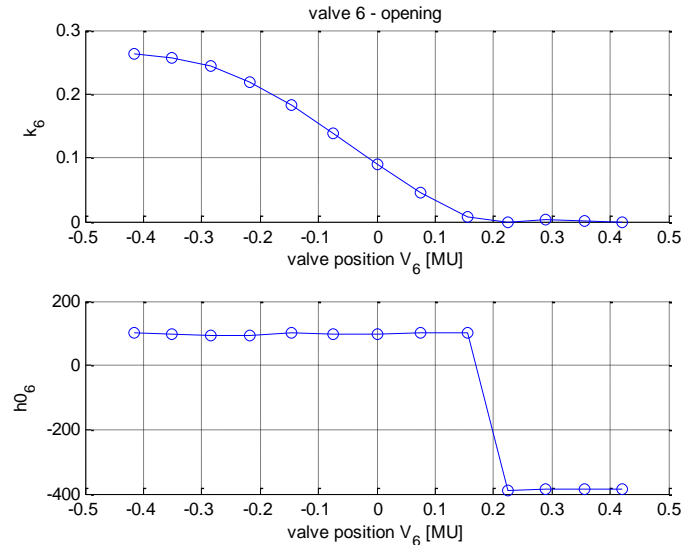


Fig. 8. Relation between valve position and  $k_6$  and  $h_{06}$

The values of  $h_{06}$  for the four rightmost experiments (i.e. valve position greater than 0.2 MU) are irrelevant because the valve was closed and no outflow was observed. The remaining experiments resulted in almost the same values of  $h_{06}$  which was expected as the values of  $h_0$  are physical values of the Amira DTS200 model.

### C. Valve flow parameter for interconnection valves

Similar approach to obtaining values of  $k$  as presented in previous subsection can be used also for valves  $V_1$  and  $V_2$  which interconnects tanks  $T_1$  and  $T_s$ , and  $T_s$  and  $T_2$ , respectively. Flow from the full tank  $T_1$  to the empty tank  $T_s$  was used to measure valve constant  $k_1$ . The other valves were closed during the experiment. According to (3), the flow can be described by two differential equations:

$$\frac{dh_1(t)}{dt} = -k_1\sqrt{|h_1(t)-h_s(t)|} \cdot \text{sign}[h_1(t)-h_s(t)] \tag{10}$$

$$\frac{dh_s(t)}{dt} = k_1\sqrt{|h_1(t)-h_s(t)|} \cdot \text{sign}[h_1(t)-h_s(t)]$$

Since the value of  $h_1$  is always higher or equal to  $h_s$ , the term inside absolute values is always nonnegative. As the water flow just from  $T_1$  to  $T_s$  and the geometry of both tanks is the same, according to mass conservation law the sum of  $h_1$  and  $h_s$  remain the same during the experiment. Then the course of draining  $T_1$  and filling  $T_s$  can be described by two independent differential equations.

$$\frac{dh_1(t)}{dt} = -k_1 \sqrt{2h_1(t) - h_\Sigma}$$

$$\frac{dh_s(t)}{dt} = -k_1 \sqrt{h_\Sigma - 2h_s(t)} \tag{11}$$

$$h_\Sigma = h_1(t) + h_s(t)$$

Solving these equations lead to time course described by second order polynomial.

$$h_1(t) = \frac{k_1^2}{2} \cdot t^2 - k_1 \sqrt{2h_1(0) - h_\Sigma} \cdot t + h_1(0)$$

$$h_s(t) = -\frac{k_1^2}{2} \cdot t^2 - k_1 \sqrt{h_\Sigma - 2h_s(0)} \cdot t + h_s(0) \tag{12}$$

An example of courses and corresponding parabolas are depicted in Fig. 9.

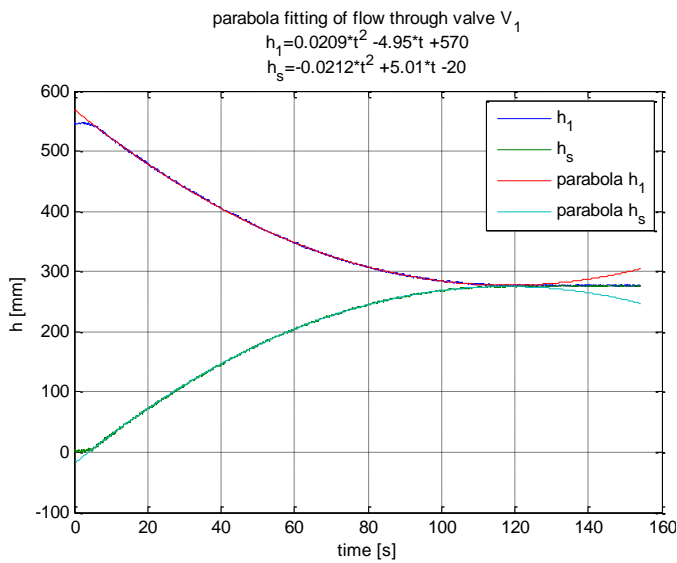


Fig. 9. Parabola fitting to the flow course through V<sub>1</sub>

Relation between position of valve V<sub>1</sub> in MU and k<sub>1</sub> is shown in Fig. 10.

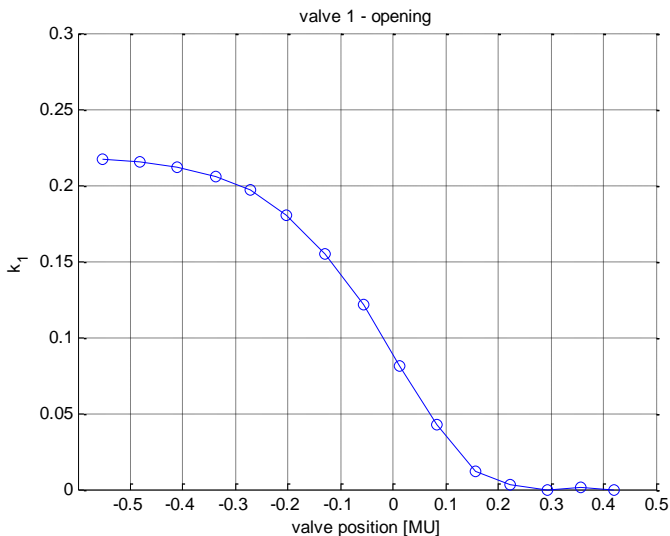


Fig. 10. Relation between valve position and k<sub>1</sub>

A similar approach as presented for valve V<sub>1</sub> was used to measure characteristics of valve V<sub>2</sub>. Resulting course of k<sub>2</sub> is depicted in Fig. 11.

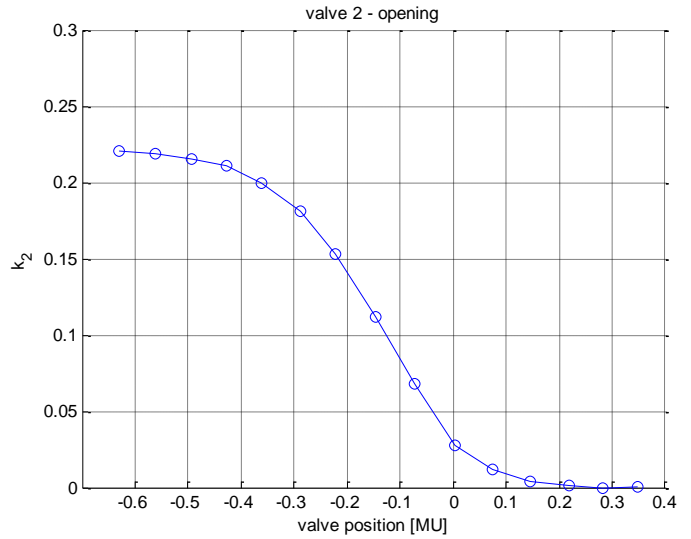


Fig. 11. Relation between valve position and k<sub>2</sub>

*D. Valve hysteresis*

The experiments presented in previous subsections were performed for opening of a valve only. At the beginning, the valve was fully closed and subsequently was partially opened to a given position. In this section a problem of closing of a valve is studied. Performed experiments are similar except initial part. The experiment starts with full tank and closed valve too, but then the valve was fully opened and then partially closed to the desired position. Therefore the same valve position (value of analogue signal from a valve) was reached but from opposite direction.

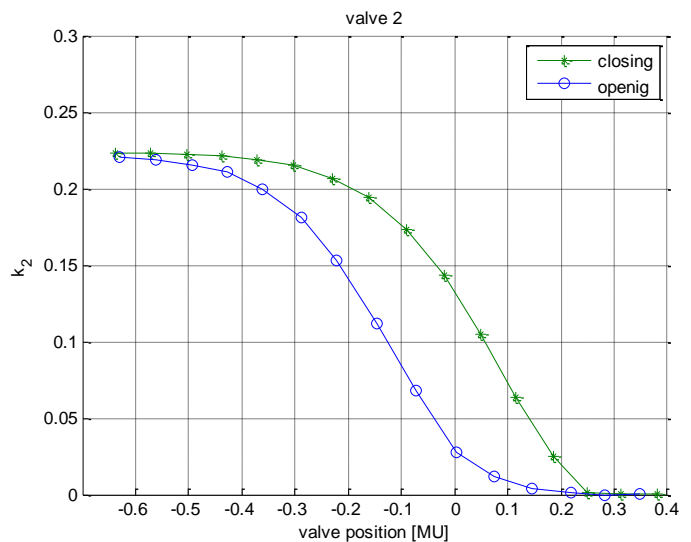


Fig. 12. Hysteresis of valve V<sub>2</sub>

Experiments unveiled a hysteresis present in all valves. The characteristics for opening and for closing of valve V<sub>2</sub> is

presented in Fig. 12. The figure shows that hysteresis plays a big role in the experiments. The value of position itself does not give sufficient information about current value of parameter  $k_2$ . For example, if the position is 0 MU the value of  $k_2$  can be anywhere in range 0.03 to 0.13. Especially in case of using the valve as an actuator the hysteresis should be taken into account. Otherwise control process can easily become unstable.

Consequently, extensive set of experiments was performed to measure hysteresis characteristics. The resulting courses of valve parameters are shown of following figures.

Three sets of experiments for opening and another three sets for closing of valve  $V_1$  were performed. Results shown in Fig. 13 provided consistent data and the hysteresis was lower comparing to Fig. 12.

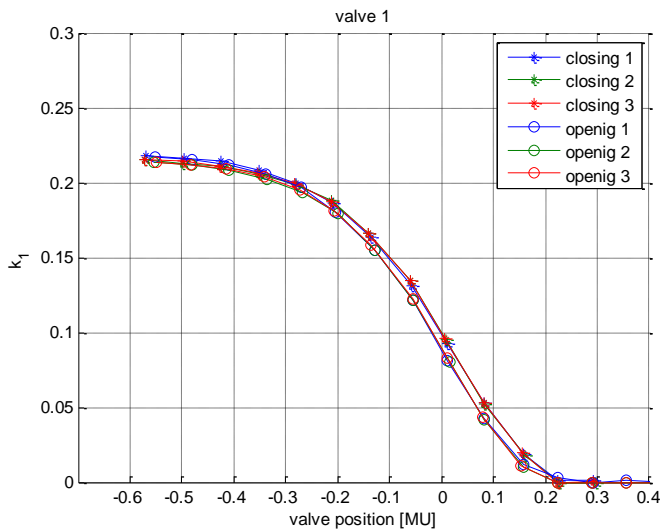


Fig. 13. Characteristics of valve  $V_1$

Four sets of experiments for closing of the valve performed in various time concerning valve  $V_2$  are depicted in Fig. 14. Individual experiments correspond to stars in the figure. Three set of experiment are presented for closing of the valve where individual experiments are represented by circles.

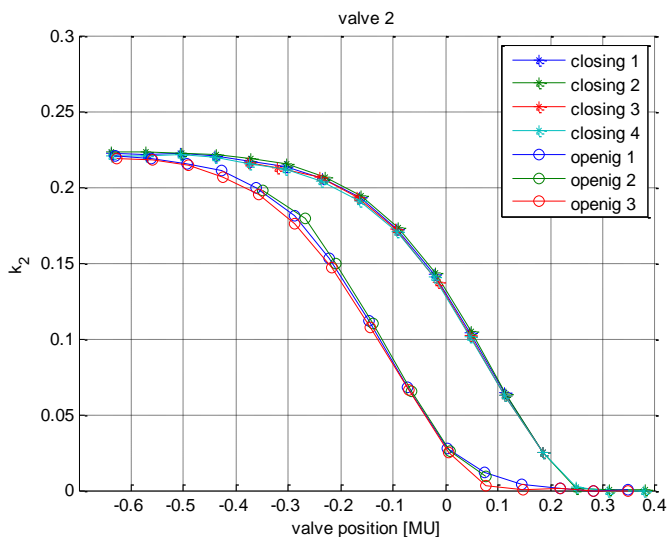


Fig. 14. Characteristics of valve  $V_2$

Similar experiments were performed also for outflow valves  $V_3$ ,  $V_4$ ,  $V_5$ , and  $V_6$ . Corresponding figures contain courses of parameter  $k$  as well as values of  $h_0$  obtained from individual experiments.

Four sets of experiments and corresponding characteristics of valve  $V_3$  is shown in Fig. 15.

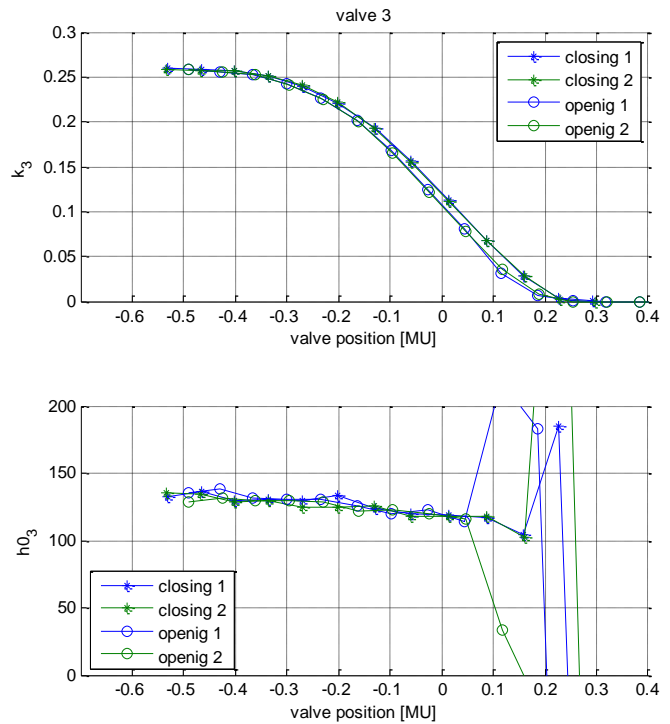


Fig. 15. Characteristics of valve  $V_3$

Twelve sets of experiments and corresponding characteristics of valve  $V_4$  is shown in Fig. 16.

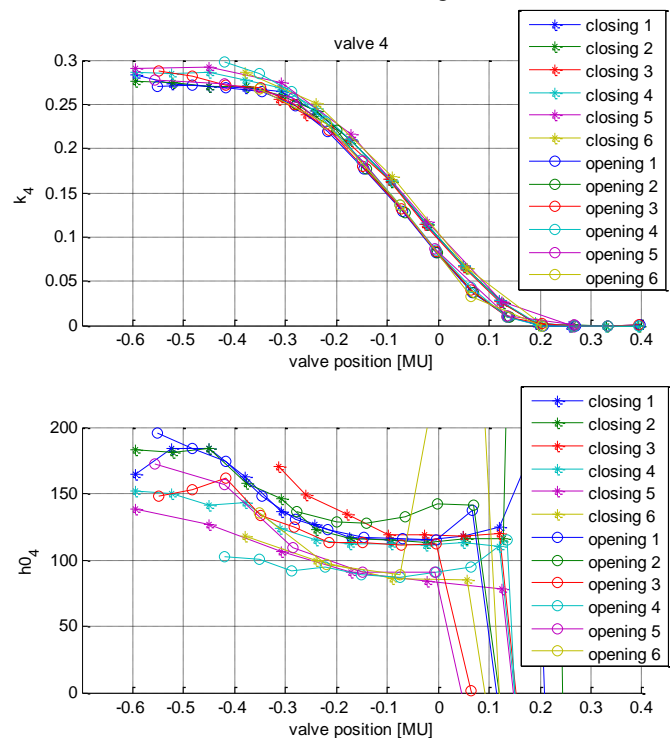


Fig. 16. Characteristics of valve  $V_4$



Five sets of experiments and corresponding characteristics of valve  $V_5$  is shown in Fig. 17.

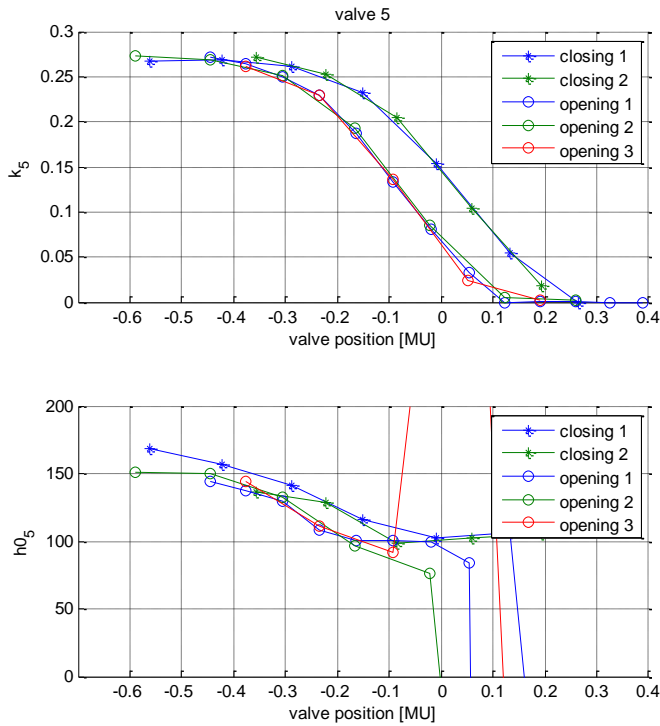


Fig. 17. Characteristics of valve  $V_5$

Five sets of experiments and corresponding characteristics of valve  $V_6$  is shown in Fig. 18.

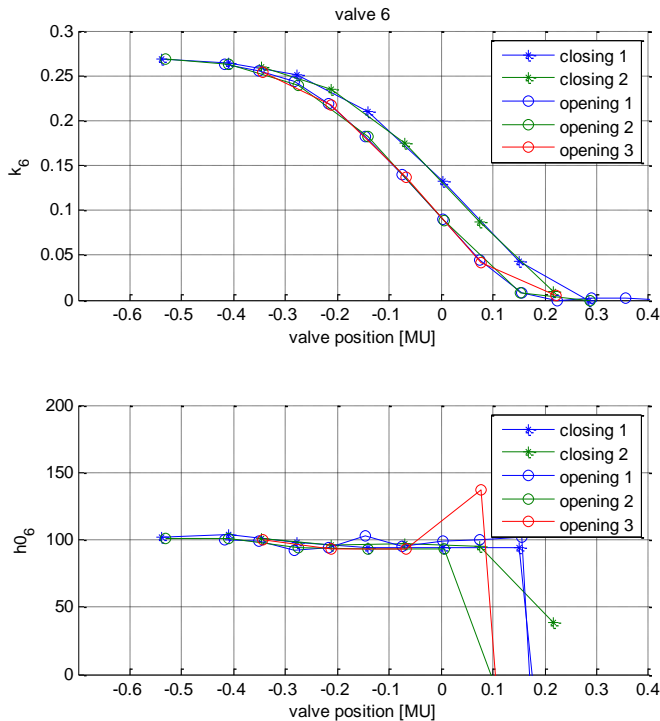


Fig. 18. Characteristics of valve  $V_6$

The hysteresis of valves  $V_1$ ,  $V_3$  and  $V_4$  is significantly smaller than the hysteresis of valves  $V_5$ ,  $V_6$  and especially

valve  $V_2$ . Experiments performed on valves  $V_3$  and  $V_6$  resulted in more consistent values of  $h_0$  when comparing with experiments performed on  $V_4$  and  $V_5$ .

Since the shape of curves corresponding to opening and closing of the valve is similar, an average difference between them in direction of position axis can be computed. This value can be used as measure of hysteresis. Values of hysteresis,  $h_0$  as well as maximal value of  $k$  for each valve is presented in Table II.

Table II  
Valve characteristics

Valve no.	hysteresis [MU]	maximal $k$	$h_0$ [mm]
1	0.0219	0.2180	-
2	0.1783	0.2237	-
3	0.0310	0.2601	126.6
4	0.0426	0.2976	127.9
5	0.1307	0.2735	121.2
6	0.0800	0.2688	97.1

E. Modeling of valve characteristics

The course of relation between valve position in MU and  $k$  is similar to step responses of dynamical system and therefore it was modeled in similar way. Other types of approximation functions, like sigmoids, were also tested, but did not achieve better results. A model based on transfer of 4<sup>th</sup> order aperiodic system produced satisfactory results. Thus relation between position and  $k$  was as follows:

$$pos < pos_0 : \quad k = k_{max} \left[ 1 - \frac{1}{6} e^{-\frac{b}{a}} \frac{b^3 + 3b^2a + 6ba^2 + 6a^3}{a^3} \right]$$

$$b = pos_0 - pos \quad (13)$$

$$pos \geq pos_0 : \quad k = 0$$

where  $pos$  is valve position in MU and parameters  $a$  and  $pos_0$  were obtained by nonlinear regression.

Values of parameters  $k_{max}$  for each valve as well as  $pos_0$  and  $a$  both for opening and closing are summed in Table III.

Table III  
Valve characteristics

Valve no.	$k_{max}$	opening		closing	
		$pos_0$	$a$	$pos_0$	$a$
1	0.2180	0.2673	0.0818	0.2892	0.0825
2	0.2237	0.1522	0.0802	0.3305	0.0807
3	0.2601	0.2689	0.0813	0.2999	0.0841
4	0.2976	0.2317	0.0880	0.2743	0.0930
5	0.2735	0.1716	0.0705	0.3024	0.0764
6	0.2688	0.2397	0.0811	0.3197	0.0856

The value of  $k_{max}$  is affected not only by valve itself but also by connection pipes. Therefore values of  $k_{max}$  are lower for interconnection valves  $V_1$  and  $V_2$  comparing to outflow valves  $V_3$ ,  $V_4$ ,  $V_5$ , and  $V_6$ .

Values of  $pos_0$  correspond to position of the curve in the valve position axis (x-axis). Therefore greater difference between  $pos_0$  for opening and for closing was achieved for valves with greater hysteresis.

Values of parameter  $a$  correspond to the slope of the curves. Higher values represent steeper slope while lower values represent softer slope.

The regression for valve  $V_2$  is presented in Fig. 19.

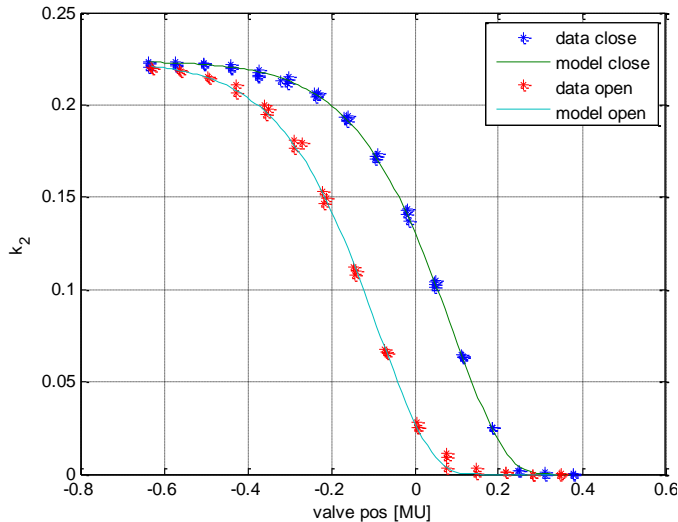


Fig. 19. Model of parameter  $k$  for valve  $V_2$

Behavior of system inside hysteresis area was studied as well. This task was time consuming because a performed set of experiments took more than 45 hours. Experiments were performed on valve  $V_2$  to study behavior inside hysteresis area. This valve was selected because its hysteresis is the greatest. Experiment results are depicted in Fig. 20. The valve was partially opened to a given position and then slightly moved towards closed position. This resulted in 7 parts, each corresponding to an initial opened position. Each of these parts contains 18 measurements for different closing.

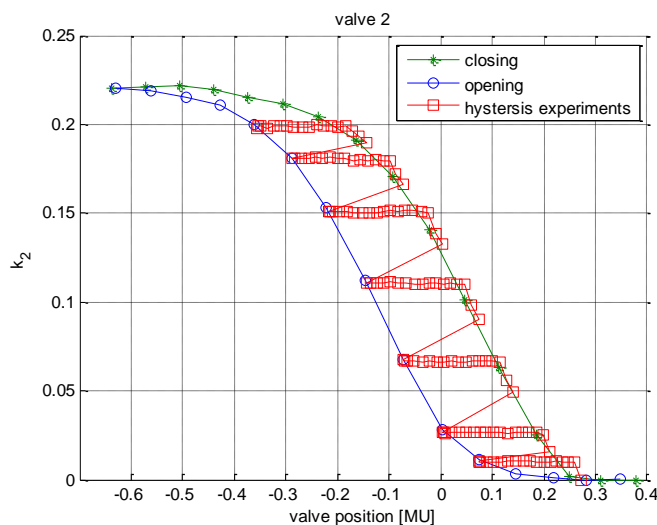


Fig. 20. Behavior inside hysteresis area (towards closed position)

It can be stated that the parameter  $k$  does not change its value till it reaches a border of hysteresis area, i.e. the curve of either opening or closing of the valve.

## V. CONCLUSION

The paper presented a development of the model of valves of a hydraulic system. The Amira DTS200 three tank system was considered but used techniques can be easily generalized to wide set of hydraulic systems. The real system contains several nonlinearities which incorporate complexity to the system. Total number of experiments concerning valves reached 433 taking altogether more than 113 hours.

Resulting model includes all major nonlinearities and can be integrated into a Simulink model of whole three tank system.

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